1940

The theory and application of tensor analysis

Robert Edwin Beam
Iowa State College

THE THEORY AND APPLICATION OF TENSOR ANALYSIS

by

Robert Edwin Beam

A Thesis Submitted to the Graduate Faculty
for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject Electrical Engineering

Approved:

Signature was redacted for privacy.

In charge of Major work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College
1940

TABLE OF CONTENTS

TABLE OF CONTENTS (CONTINUED)

TABLE OF CONTENTS (CONTINUED)

I. INTRODUCTION

A civilized man seems to have an habitual craving for order, unity, deeper insight, and minimum labor. He attempts to generalize the results of investigations and to establish a correspondence between them and mental processes. In the case of investigations of physical phenomena he utilizes the isomorphic relationships between mental processes, geometrical representation, and algebraic formulas.

In almost any attempt to establish a correspondence between physical phenomena and mental processes, the principles of geometrical representation of algebraic formulas are used. Almost immediately, the question of whether a formulation in terms of coordinates expresses something which characterizes the physical occurrence or the coordinate system arises. A sufficient condition which, if fulfilled, insures that a statement has a meaning which is independent of the reference coordinate system is desired. This is what tensor analysis attempts to do. The method of tensors first assigns coordinates, and then shows how to obtain results which, though expressed in terms of coordinates, do not depend upon the choice of them. Thus tensor analysis provides a suitable mathematical tool for the establishment of a correspondence between physical phenomena and mental processes, but its application to differential geometry provides the method of reasoning to be used.

The objectives of this thesis are to present some of the fundamental principles of tensor analysis and differential geometry, and to apply these principles to show: (1) how vector analysis is related to tensor analysis; (2) how Lagrange's dynamical equations can be given a geometrical interpretation; (3) how to transform the quantities in Lagrange's equations from Cartesian reference coordinates to general curvilinear reference coordinates, and thence to an ennuple of congruences; (4) how to apply the principles of transformation of Lagrange's equations to the analysis of rotating electrical machinery.

II. TENSOR ANALYSIS

The fundamental idea forming the basis of the absolute differential calculus, popularly known as tensor analysis, is that of changing variables or coordinates. Early study in connection with tensor analysis was centered about an invariant quadratic form. The general theory of the quadratic differential form was inaugurated by the work of Riemann which was read before the Philosophical Faculty of the University of Goettingen in 1854. Riemann's work was published in his "Gesammelte Werke" (19). Before the details of Riemann's work were known, the main lines of the theory were developed by Lipshitz (14) and Christoffel (4), both of whom found the components of the affine connection and the curvature tensor; Christoffel also used covariant differentiation. Riemann's work demonstrated that the Euclidean geometry, the "hyperbolic" geometry of Bolyai and Lobatchevsky, and his "elliptic" geometry were special cases of a metric geometry based upon the invariant quadratic differential form $ds^2 = g_{ij}\, ds^i\, dx^j$.

The name covariant differentiation and the discovery of its importance was due to Ricci. His studies and those of Levi-Civita were set forth in a condensed form in a joint paper (18). It was Ricci who introduced also the use of subscripts and superscripts to distinguish between two different laws of transformation known as covariant and contravariant.

The invariant theory of tensor analysis underlies the modern differential geometry of infinitesimal displacements. In this field some non-tensor invariants, such as the affine connection, have assumed great importance. Since geometry serves as a mode of reasoning, it is considered desirable to introduce at least some of the more advanced tensor and non-tensor invariant objects and relations in geometry. Nevertheless, the invariant theory of tensor analysis exists as a branch of analysis, completely differentiated from geometry.

The name tensor was introduced by Einstein in his theory of general relativity in the year 1916 (6). What was previously called the absolute differential calculus soon afterwards became popularly known as tensor analysis. Both Einstein (7, Chapt. I) and Eddington (5, p. 44) have written good accounts of the physical significance of tensors.

Until the works of Weyl (24, 25), Eddington (5, p. 213), Schouten (20), and Carten (3), differential geometry had been mostly limited to the invariant theory based on the fundamental quadratic form of Riemann. It was Weyl who first generalized the idea of the early workers by introducing general functions of the coordinates as symmetric affine connections in place of the Christoffel symbols and by limiting parallel displacements to infinitesimal distances. Eddington, Schouten, and Carten have generalized Weyl's method to include asymmetric connections; the resulting geometry is often called non-Riemannian geometry.

There are two different types of transformation coefficients which may occur in tensor theory. The older tensor theory was limited to coordinate transformations where tensor components could be considered with reference to the differentials of coordinates. In this case the transformation coefficients were partial derivatives of the coordinate variables concerned, the conditions for the existence of exact differentials or the integrability conditions being satisfied, so that a linear transformation of the coordinate differentials, for example, could be integrated to obtain the coordinate variables. Such transformations are sometimes called holonomous transformations. There is, however, a more general type of transformation in which the integrability conditions for an exact differential need not be satisfied. In this case the order of partial differentiation cannot be interchanged and the same result obtained; an asymmetric connection results. Such transformations are sometimes called non-holonomous transformations. An optional viewpoint to that of transformation of non-holonomous parameters consists of considering the tensor components defined with respect to differentials of arc, and transformed from one set of differentials of arc to another set, without requiring the existence of the underlying variables of the differentials of arc. This latter tensor theory is known as intrinsic tensor analysis. In 1934, Graustein's paper on "The Geometry of Riemannian Spaces" was published (8); he gave an excellent account of the intrinsic tensor theory.

It should now be quite clear that transformations of coordinate variables and of quantities represented in terms of these variables are the chief concern of tensor analysis; therefore it seems desirable to begin a treatment of tensor analysis with the fundamentals of transformation of coordinates.

1. Transformation of coordinates

A point (a group of n ordered real numbers) in an n-dimensional manifold (a continuous arrangement or set of points) may be thought of as being represented by any n independent variables x^i, where i takes the values 1 to n, representing the coordinates of the point. (Unless stated otherwise the coordinates will be considered real.)

If n real functions φ^i of the variables x^i satisfy the Jacobian relationship

$$\left|\frac{\partial \varphi^i}{\partial x^j}\right| \neq 0, \tag{2.1}$$

the functions are said to be independent. Then, if

$$\overline{x}^i = \varphi^i (x^1, x^2, \ldots x^n) \tag{2.2}$$

the n quantities $\overline{x}^i$ are another set of coordinates of the n-manifold; when the coordinates x^i of any point P are substituted in equation (2.2), these equations give the $\overline{x}^i$ coordinates of P. The equations (2.2) define a transformation of coordinates of the n-manifold; if these equations are linear functions of the coordinates, they are said to define an affine transformation.

Since the equations (2.2) are independent, they can be solved simultaneously for the x^i in terms of the $\overline{x}^i$. This operation yields the n equations

$$x^i = \psi^i(\overline{x}^1, \overline{x}^2, \ldots \overline{x}^n), \tag{2.3}$$

which represent transformations from the $\bar{x}^i$ to the x^i coordinates.

2. Kronecker deltas and the summation convention

If the x^i as functions of the ?j $\bar{x}^i$ equation (2.3) are differentiated with respect to a particular coordinate x^j, there is obtained

$$\frac{\partial x^k}{\partial x^j} = \frac{\partial x^k}{\partial \bar{x}^\alpha} \cdot \frac{\partial \bar{x}^\alpha}{\partial x^j} \qquad \alpha = 1, \ldots, n, \tag{2.4}$$

where the indices k and j indicate particular x^i terms, and the repeated α index indicates a summation on α. α is known as a "dummy" index; it may be changed at will to any other literal index. This use of a repeated index to indicate a summation is known as the summation convention; unless otherwise stated it will be used on the succeeding pages without further mention.

Since the x^i are independent, equations (2.4) are either equal to unity or zero depending upon whether i is equal or not equal to j; that is

$$\frac{\partial x^k}{\partial \bar{x}} \cdot \frac{\partial \bar{x}}{\partial x^{i\,j?}} = \delta^k_j \qquad \begin{array}{l} \delta^k_j = 1 \text{ if } k = j \\ \delta^k_j = 0 \text{ if } k \neq j. \end{array} \tag{2.5}$$

These are called the Kronecker deltas; they may be generalized for k subscripts and k superscripts and denoted by (22, p. 3)

$$\delta^{i_1\ i_2\ \ldots\ i_k}_{j_1\ j_2\ \ldots\ j_k}. \tag{2.6}$$

If the superscripts of (2.6) are all different, and the subscripts are the same set of numbers, the value of the symbol is 1 or -1 according to whether an even or an odd permutation is required to arrange the superscripts in the same order as the subscripts; in all other cases its value is zero.

In a similar manner

$$\frac{\partial \bar{x}^k}{\partial x^a} \cdot \frac{\partial x^a}{\partial \bar{x}^j} = \delta^k_j. \tag{2.7}$$

3. <u>Transformation matrices</u>

If k in equation (2.7) is held constant and j is allowed to assume successively the values 1 to n, n linear equations in $\frac{\partial \bar{x}^k}{\partial x^i}$ result. These equations may be solved by Cramer's rule; this yields

$$\frac{\partial \bar{x}^k}{\partial x^a} = \frac{\text{cofactor of } \frac{\partial x^a}{\partial \bar{x}^k} \text{ in } \left|\frac{\partial x}{\partial \bar{x}}\right|}{\left|\frac{\partial x}{\partial \bar{x}}\right|}. \tag{2.8}$$

If k and α in (2.8) are allowed to assume successively all their possible values by letting α assume all its possible values for each value of k, the following matrix can be formed:

$$\left[\frac{\partial \bar{x}^k}{\partial x^a}\right] = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \cdots\cdots & \frac{\partial \bar{x}^1}{\partial x^n} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \cdots\cdots & \frac{\partial \bar{x}^2}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \bar{x}^n}{\partial x^1} & \frac{\partial \bar{x}^n}{x^2} & \cdots\cdots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix}. \tag{2.9}$$

This matrix is known as the holonomous transformation matrix; it is used in the definition of a tensor.

Similarly, equations (2.8) might have been solved for the inverse functions $\dfrac{\partial x^a}{\partial \bar{x}^k}$ and the matrix

$$\left[\frac{\partial x^a}{\partial \bar{x}^k}\right] = \begin{bmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1} & \dfrac{\partial x^1}{\partial \bar{x}^2} & \cdots\cdots & \dfrac{\partial x^1}{\partial \bar{x}^n} \\ \dfrac{\partial x^2}{\partial \bar{x}^1} & \dfrac{\partial x^2}{\partial \bar{x}^2} & \cdots\cdots & \dfrac{\partial x^2}{\partial \bar{x}^n} \\ \dfrac{\partial x^n}{\partial \bar{x}^1} & \dfrac{\partial x^n}{\partial \bar{x}^2} & \cdots\cdots & \dfrac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix} \qquad (2.10)$$

obtained.

4. The "group" property

It should be noted that the transformation matrices of equations (2.9) and (2.10) belong to a group; that is, the matrices and the operations to which they are subjected satisfy the following four conditions:

(1) The product entity C of two entities A B belongs to the same group.

(2) The products of several elements obey the associative law of multiplication.

$$A \circ B \circ C = A \circ (B \circ C) = (A \circ B) \circ C, \text{ etc.}$$

(3) One element of the group is the "unit element" so that multiplication by it leaves any element unchanged.

(4) Each element has an "inverse element" so that the product of an element and its inverse is the unit element.

Thus the whole theory of groups is immediately applicable to the transformation matrices; this makes possible the subdivision of complex problems into several simpler problems.

5. Invariants

According to Veblen (22, p. 14) "an object of any sort which is not changed by transformation of coordinates is called an invariant." As examples, any point, any set of points, and any point function are invariants.

If a point function is represented by $\varphi(x^i)$ in an x^i coordinate system, and by $\varphi(\bar{x}^i)$ in any $\bar{x}^i$ coordinate system, the point function is called an absolute scalar, clearly an invariant.

In succeeding sections a more general class of invariants, having several components in each coordinate system, will be considered.

6. Contravariant vectors

? The n differentials $d\bar{x}^i$ of the n coordinate variables $\bar{x}^i$, the $\bar{x}^i$ being functions of any n independent variables x^i, are given by the equations

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^\alpha} dx^\alpha, \qquad (2.11)$$

neglecting infinitesimals of higher order. Since these equations are linear in the differentials, any transformation of coordinates is locally linear.

The abstract object which is determined by equations (2.11), and whose value is independent of the reference coordinate system is called a contravariant vector; the n differentials are called its components. In general, the abstract object whose n components $V^j(x^i)$ transform from the x^i system to the $\bar{x}^i$ system by the rule

$$\bar{V}^k(\bar{x}^i) = \frac{\partial \bar{x}^k}{\partial x^\alpha} \; V^\alpha(x^i), \tag{2.12}$$

where the $\frac{\partial \bar{x}^k}{\partial x^\alpha}$ are components of the transformation matrix of equation (2.9), is defined to be a contravariant vector; the raised position of the indentification postscript on a component is used to indicate the contravariant character of the component.

7. Covariant vectors

The n partial derivatives of a scalar point function $\varphi(x^i)$ with respect to the n variables $\bar{x}^i(x^i)$ are given by the rule of partial differentiation

$$\frac{\partial \varphi}{\partial \bar{x}^i} = \frac{\partial \varphi}{\partial x^\alpha} \cdot \frac{\partial x^\alpha}{\partial \bar{x}^i}, \tag{2.13}$$

where terms of higher order than the first in the infinitesimals are neglected. These partial derivatives represent the components of the gradient of φ.

The abstract object which is determined by equations (2.13), and which is independent of the reference system is called a covariant vector; the partial derivatives are its components. In general, the abstract object x whose n components $U_i(x^i)$ transform from the x^i system to any $\bar{x}^i$ system by the rule

$$\bar{U}_i(\bar{x}^i) = U_\alpha(x^i) \, \frac{\partial x^\alpha}{\partial \bar{x}^i} \tag{2.14}$$

is defined to be a covariant vector; the lowered position of the identification postscript on a component is used to indicate the covariant character of the component.

8. Definition of a general tensor

The abstract invariant object whose components in a coordinate system x^i are denoted by

$$T^{a_1 a_2 \cdots a_r}_{b_1 b_2 \cdots b_s}, \tag{2.15}$$

where each of the indices take successively the values 1 to n and yield $n^{(r+s)}$ components, and whose law of transformation is

$$\overline{T}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = T^{a_1 a_2 \cdots a_r}_{b_1 b_2 \cdots b_s} \frac{\partial \overline{x}^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{a_r}} \frac{\partial x^{b_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{b_s}}{\partial \overline{x}^{j_s}}, \tag{2.16}$$

is defined to be a mixed tensor of order r + s. The tensor T is contravariant in r indices and covariant in s indices, as indicated by the positions of the indices.

9. Symmetric and anti-symmetric tensors

Ordinarily, formula (2.16) yields $n^{(r+s)}$ independent components. Sometimes, however, a portion of these components are equal in absolute value and show a definite symmetry in two or more of their indices of the same variance character. It is customary to distinguish between two cases: (a) symmetric; (b) anti-symmetric.

The symmetric case implies the equality of all the components deduced from the indices of symmetry by any permutation of these indices. For example, the fourth order symmetric contravariant tensor T has components obeying the following relationships:

$$T^{ijkl} = T^{lkji} = T^{jikl}, \text{ etc.};$$
$$T^{iijk} = T^{jkii} = T^{jiik}, \text{ etc.} \qquad (2.17)$$

In the general anti-symmetric case involving m indices of anti-symmetry, a more careful examination is required. A chosen permutation, such as 123···m, must be used as a reference permutation; all the permutations deduced from this reference permutation by an even number of exchanges of two indices are called even permutations; all the permutations deduced from this reference permutation by an odd number of exchanges of two indices are called odd permutations. Complete anti-symmetry requires that even permutations conserve the sign of the components, and that odd permutations reverse the sign; components with two identical indices are consequently zero. For example, the fourth order anti-symmetric contravariant tensor T has components obeying the following rules:

$$T^{ijkl} = T^{lkji} = -T^{jikl} = -T^{ijlk} = -T^{ikjl}; \qquad (2.18)$$

$$T^{iiii} = T^{jjjj} = 0, \text{ etc.} \qquad (2.19)$$

10. Tensor algebra

Algebraic combinations of tensors make possible the construction of further tensors from given tensors; therefore it is reasonable to expect these principles of combinations to be important in application.

The sum of two tensors of the same variance character is formed by adding corresponding components. If R and S are two tensors of the same order p = (m + n) and the same variance character, then their sum is given by

$$T^{a_1 \cdots a_m}_{b_1 \cdots b_n} = R^{a_1 \cdots a_m}_{b_1 \cdots b_n} + S^{a_1 \cdots a_m}_{b_1 \cdots b_n}. \tag{2.20}$$

The tensor character of this operation is easily demonstrated by a transformation of the components to a different coordinate system. If in the new coordinate system $\bar{x}^i$ a component of T is represented by $\bar{T}^{i_1 \cdots i_m}_{j_1 \cdots j_n}$, then

$$\begin{aligned} \bar{T}^{i_1 \cdots i_m}_{j_1 \cdots j_n} &= \frac{\partial \bar{x}^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{a_m}} \frac{\partial x^{b_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{b_n}}{\partial \bar{x}^{j_n}} R^{a_1 \cdots a_m}_{b_1 \cdots b_n} \\ &+ \frac{\partial \bar{x}^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{a_m}} \frac{\partial x^{b_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{b_n}}{\partial \bar{x}^{j_n}} S^{a_1 \cdots a_m}_{b_1 \cdots b_n} \\ &= \frac{\partial \bar{x}^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{a_m}} \frac{\partial x^{b_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{b_n}}{\partial \bar{x}^{j_n}} \left(R^{a_1 \cdots a_m}_{b_1 \cdots b_n} + S^{a_1 \cdots a_m}_{b_1 \cdots b_n} \right) \end{aligned} \tag{2.21}$$

This confirms the tensor character of the sum.

The <u>contraction</u> of a tensor can be applied to any mixed tensor or its equivalent. It is formed by setting a contravariant index equal to a covariant index and performing the indicated summation. Contraction reduces the order of the tensor by two. The tensor character of this operation is easily demonstrated by starting with the definition of a general mixed tensor

$$\bar{T}^{i_1 i_2 \cdots i_m}_{j_1 j_2 \cdots j_n} = \frac{\partial \bar{x}^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{a_m}} \frac{\partial x^{b_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{b_n}}{\partial \bar{x}^{j_n}} T^{a_1 \cdots a_m}_{b_1 \cdots b_n}. \tag{2.22}$$

A contraction with respect to two indices of opposite variance character is then made. This operation yields

$$\bar{T}^{i_1\, i_2 \cdots\, i_m}_{i_1\, j_2 \cdots\, j_n} = \frac{\partial \bar{x}^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{a_m}} \frac{\partial x^{b_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{b_n}}{\partial \bar{x}^{j_n}} T^{a_1 \cdots\, a_m}_{b_1 \cdots\, b_n}$$

$$= \delta^{b_1}_{a_1} \frac{\partial \bar{x}^{i_2}}{\partial x^{a_2}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{a_m}} \frac{\partial x^{b_2}}{\partial \bar{x}^{i_2}} \cdots \frac{\partial x^{b_n}}{\partial \bar{x}^{j_n}} T^{a_1 \cdots\, a_m}_{b_1 \cdots\, b_n}$$

$$= \frac{\partial \bar{x}^{i_2}}{\partial x^{a_2}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{a_m}} \frac{\partial x^{b_2}}{\partial \bar{x}^{i_2}} \cdots \frac{\partial x^{b_n}}{\partial \bar{x}^{j_n}} T^{a_1 \cdots\, a_m}_{a_1 \cdots\, b_n} \qquad (2.23)$$

hence

$$T^{i_1\, i_2 \cdots\, i_m}_{i_1\, j_2 \cdots\, j_n} = R^{i_2 \cdots\, i_m}_{j_2 \cdots\, j_n} \qquad (2.24)$$

a tensor of order (m+n-2); therefore the demonstration is completed.

The <u>product</u> of two tensors of any variance character has by definition the components

$$V^{i_1 \cdots\, i_k i_l \cdots\, i_q}_{j_1 \cdots\, j_n j_o \cdots\, j_s} = T^{i_1 \cdots\, i_k}_{j_1 \cdots\, j_n} U^{i_l \cdots\, i_q}_{j_o \cdots\, j_s} \qquad (2.25)$$

where V is the product tensor, and T and U are the component tensors. It is obvious that the resulting tensor V is a tensor of order equal to the sum of the orders of T and U. This rule provides a method of forming tensors of higher order from vectors or tensors of lower order.

11. <u>The quotient law</u>

A multilinear form is considered to be an invariant expression in several variables and to be representable, in general, by

$$\sum_{\substack{i_1 \cdots\, i_m \\ j_1 \cdots\, j_r}}^{n} A(i_1 \cdots i_m j_1 \cdots j_r)\, T^{i_1} \cdots T^{i_m}\, U_{j_1} \cdots U_{j_r}. \qquad (2.26)$$

The coefficients of such an invariant form in the variables $T^{i_1}, \dots T^{i_m}$, and $U_{j_1}, \dots U_{j_r}$ form a tensor with a variance character opposite to that of the product of the variables; this statement is known as the quotient law.

Since equation (2.26) represents an invariant, it has the same value in any coordinate system. In order to show that the coefficients $A(i_1 \dots i_m j_1 \dots j_r)$ are tensor components, it is only necessary to show that the coefficients transform in the same way as the components of a tensor. If the multilinear form expressed in the variables x^i is equated to the multilinear form expressed in the variables $\overline{x}^i$ so that

$$\sum_{\substack{i_1 \dots i_m \\ j_1 \dots j_r}} A(i_1 \dots i_m j_1 \dots j_r)\, T^{i_1} \dots T^{i_m} U_{j_1} \dots U_{j_r} = \sum_{\substack{k_1 \dots k_m \\ l_1 \dots l_r}} \overline{A}(k_1 \dots k_m l_1 \dots l_r) \overline{T}^{k_1} \dots \overline{T}^{k_m} \overline{U}_{l_1} \dots \overline{U}_{l_r}, \qquad (2.27)$$

and if the new variables are expressed in terms of the old by means of the usual transformation formulas so that

$$\sum A.T^{i_1} \dots T^{i_m} U_{j_1} \dots U_{j_r} = \sum \overline{A}.\frac{\partial \overline{x}^{k_1}}{\partial x^{i_1}} T^{i_1} \dots \frac{\partial \overline{x}^{k_m}}{\partial x^{i_m}} T^{i_m} \frac{\partial x^{j_1}}{\partial \overline{x}^{l_1}} U_{j_1} \dots \frac{\partial x^{j_r}}{\partial \overline{x}^{l_r}} U_{j_r}, \qquad (2.28)$$

then by equating coefficients it is clear that

$$A = \overline{A} \frac{\partial \overline{x}^{k_1}}{\partial x^{i_1}} \dots \frac{\partial \overline{x}^{k_m}}{\partial x^{i_m}} \frac{\partial x^{j_1}}{\partial \overline{x}^{l_1}} \dots \frac{\partial x^{j_r}}{\partial \overline{x}^{l_r}}. \qquad (2.29)$$

Equation (2.29) is the same as the law of transformation of the components of a tensor with r contravariant and m covariant indices; therefore the A's are components of a mixed tensor

$$A^{j_1 \cdots j_r}_{i_1 \cdots i_m} = \bar{A}^{l_1 \cdots l_r}_{k_1 \cdots k_m} \frac{\partial \bar{x}^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{k_m}}{\partial x^{i_m}} \frac{\partial x^{j_1}}{\partial \bar{x}^{l_1}} \cdots \frac{\partial x^{j_r}}{\partial \bar{x}^{l_r}} \tag{2.30}$$

of order (r+m), thus completing the demonstration. This quotient law is very useful as a criterion of tensor character.

12. The decomposition of a tensor into symmetric and anti-symmetric parts

All tensors are either symmetric, anti-symmetric, or a combination of the two with respect to any two indices of the same variance character. This can be demonstrated by making

$$S^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} = \frac{1}{2}\left(T^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} + T^{a_2 a_1 \cdots a_n}_{b_1 b_2 \cdots b_p}\right) \tag{2.31}$$

a component of a tensor S, symmetric with respect to two indices a_1 and a_2, and by making

$$A^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} = \frac{1}{2}\left(T^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} - T^{a_2 a_1 \cdots a_n}_{b_1 b_2 \cdots b_p}\right) \tag{2.32}$$

a component of a tensor A, anti-symmetric with respect to the same indices; then by adding the respective components it follows that

$$S^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} + A^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} = \frac{1}{2}\left(T^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} + T^{a_2 a_1 \cdots a_n}_{b_1 b_2 \cdots b_p}\right)$$

$$+ \frac{1}{2}\left(T^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p} - T^{a_2 a_1 \cdots a_n}_{b_1 b_2 \cdots b_p}\right) = T^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_p}; \tag{2.33}$$

thus the demonstration is completed.

13. Exterior products

The components of r n-dimensional vectors ($r < n$) having the same variance character can be arranged to form the following matrix (2, p. 41)

$$\begin{bmatrix} a^1 & a^2 & \dots & a^n \\ b^1 & b^2 & \dots & b^n \\ \dots & \dots & \dots & \dots \\ r^1 & r^2 & \dots & r^n \end{bmatrix} \qquad (2.34)$$

From this matrix a completely anti-symmetric tensor, known as the exterior product of the vectors, can be formed by:

(1) Choosing a group of r numerals $\alpha \beta \gamma \dots \rho$ from among the numerals $1, 2, \dots n$, with or without repetitions

(2) Forming the determinant

$$\begin{vmatrix} a^\alpha & a^\beta & \dots & a^\rho \\ b^\alpha & b^\beta & \dots & b^\rho \\ \dots & \dots & \dots & \dots \\ r^\alpha & r^\beta & \dots & r^\rho \end{vmatrix}$$

which has for its first column the α column of the table; for its second column the column β ; etc.

(3) Setting the determinant thus formed equal to a component of the tensor T, where

$$T^{\alpha\beta\dots\rho} = \begin{vmatrix} a^\alpha & a^\beta & \dots & a^\rho \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ r^\alpha & r^\beta & \dots & r^\rho \end{vmatrix} \qquad (2.35)$$

The two well known theorems of determinant theory -- an interchange of two rows (or columns) of a determinant reverses the algebraic sign of the determinant, and a determinant with two equal rows (or columns) is nul — demonstrate the completely anti-symmetric character of the exterior product. The exterior products are sometimes called <u>multivectors</u>.

14. Pseudo-tensors

True anti-symmetric tensors present a restricted number of independent components. A reduction of the number of indices can be made; thus a simplification in some formulas is accomplished. The reduced quantities are not tensors because they do not transform as tensors; they are called pseudo-tensors (2, p. 45).

There are two principal types of pseudo-tensors; namely, tensor densities and tensor capacities. These two types will be given some consideration on the succeeding pages.

Anti-symmetric tensors of the exterior product type [Eq. (2.35)] have all components with either the same absolute value or zero; this fact makes possible the formation of the two most important special types of pseudo-tensors; namely, scalar densities and scalar capacities.

15. Scalar densities

The absolute value of the existent components of an nth order completely anti-symmetric covariant tensor C of the exterior product type can be represented by $|\gamma|$, where the positive value of γ is given by

$$\gamma = C_{12\ldots m} \quad (123\ldots n \text{ is the reference order}) \qquad (2.36)$$

This quantity γ can be shown to possess a single component; but it undergoes a change of value when the reference axes are changed. This change in value of γ under a transformation of coordinates will now be demonstrated.

The transformation formula of the components of the tensor C is given by

$$C_{\alpha\beta\ldots\varsigma} = \frac{\partial \overline{x}^a}{\partial x^\alpha} \cdots \frac{\partial \overline{x}^m}{\partial x^\varsigma} \overline{C}_{ab\ldots m}. \tag{2.37}$$

Now

$$\overline{C}_{ab\ldots m} = +\overline{\gamma} \quad \text{(if ab...m is an even permutation of 12...m.)} \tag{2.38}$$

and

$$\overline{C}_{ab\ldots m} = -\overline{\gamma} \quad \text{(if an odd permutation).} \tag{2.39}$$

When these values are substituted in equation (2.39), the following result is obtained:

$$\gamma = C_{12\ldots m} = \sum_{ab\ldots m=1}^{n} \pm \frac{\partial \overline{x}^a}{\partial x^1} \frac{\partial \overline{x}^b}{\partial x^2} \cdots \frac{\partial \overline{x}^m}{\partial x^m} \overline{\gamma}. \tag{2.40}$$

The sum of the terms $\pm \frac{\partial \overline{x}^a}{\partial x^1} \cdots \frac{\partial \overline{x}^m}{\partial x^n}$ is made with a + sign or a - sign following whether the permutation is even or odd; it is the same as the definition of the determinant of $\frac{\partial \overline{x}^i}{\partial x^j}$. It then follows that

$$\gamma = \begin{vmatrix} \frac{\partial \overline{x}^1}{\partial x^1} & \frac{\partial \overline{x}^1}{\partial x^2} & \cdots & \frac{\partial \overline{x}^1}{\partial x^n} \\ \frac{\partial \overline{x}^2}{\partial x^1} & \frac{\partial \overline{x}^2}{\partial x^2} & \cdots & \frac{\partial \overline{x}^2}{\partial x^n} \\ \frac{\partial \overline{x}^n}{\partial x^1} & \frac{\partial \overline{x}^n}{\partial x^2} & \cdots & \frac{\partial \overline{x}^n}{\partial x^n} \end{vmatrix} \overline{\gamma} = \Delta^{-1} \overline{\gamma}; \tag{2.41}$$

hence the demonstration is completed. γ is called a scalar density.

16. Scalar capacities

Scalar capacities are formed in the same way as are scalar densities, the only difference being that the starting anti-symmetric tensor is contravariant in its indices; thus a scalar capacity is given by

$$\tau = \Delta \bar{\tau} \,, \tag{2.42}$$

where τ is the scalar capacity and Δ is the inverse of the determinant of equation (2.41).

17. Tensor densities

A tensor density is defined to be the product of any tensor and a scalar density; for example,

$$\Omega^{a_1 \cdots a_m}_{b_1 \cdots b_n} = \gamma T^{a_1 \cdots a_m}_{b_1 \cdots b_n} \tag{2.43}$$

where γ is a scalar density and $T^{a_1 \cdots a_m}_{b_1 \cdots b_n}$ is a component of any tensor, is a tensor density, and not a true tensor. Some authors, however, define tensors in such a way that they can include as a weighting factor a scalar density or capacity (22, p. 22).

The formula of transformation of $\Omega^{a_1 \cdots a_m}_{b_1 \cdots b_n}$ is given by

$$\begin{aligned} \Omega^{a_1 \cdots a_m}_{b_1 \cdots b_n} &= \Delta^{-1} \bar{\gamma} \frac{\partial \bar{x}^{j_1}}{\partial x^{b_1}} \cdots \frac{\partial \bar{x}^{j_n}}{\partial x^{b_n}} \frac{\partial x^{a_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{a_m}}{\partial \bar{x}^{i_m}} \bar{T}^{i_1 \cdots i_m}_{j_1 \cdots j_n} \\ &= \Delta^{-1} \frac{\partial \bar{x}^{j_1}}{\partial x^{b_1}} \cdots \frac{\partial \bar{x}^{j_n}}{\partial x^{b_n}} \frac{\partial x^{a_1}}{\partial \bar{x}^{i_1}} \cdots \frac{\partial x^{a_m}}{\partial \bar{x}^{i_m}} \bar{\Omega}^{i_1 \cdots i_m}_{j_1 \cdots j_n}. \end{aligned} \tag{2.44}$$

where

$$\overline{\Omega}^{i_1 \cdots i_m}_{j_1 \cdots j_n} = \overline{\gamma}\, \overline{T}^{i_1 \cdots i_m}_{j_1 \cdots j_n}. \tag{2.45}$$

This transformation proves that Ω is not a true tensor.

18. Tensor capacities

A tensor capacity is defined as the product of a scalar capacity and any tensor. If

$$\omega^{a_1 \cdots a_m}_{b_1 \cdots b_n} = \tau T^{a_1 \cdots a_m}_{b_1 \cdots b_n} \tag{2.46}$$

represents such a tensor capacity, its rule of transformation is

$$\begin{aligned}\omega^{a_1 \cdots a_m}_{b_1 \cdots b_n} &= \Delta \overline{\tau} \frac{\partial \overline{x}^{j_1}}{\partial x^{b_1}} \cdots \frac{\partial \overline{x}^{j_n}}{\partial x^{b_n}} \frac{\partial x^{a_1}}{\partial \overline{x}^{i_1}} \cdots \frac{\partial x^{a_m}}{\partial \overline{x}^{i_m}} \overline{T}^{i_1 \cdots i_m}_{j_1 \cdots j_n} \\ &= \Delta \frac{\partial \overline{x}^{j_1}}{\partial x^{b_1}} \cdots \frac{\partial \overline{x}^{j_n}}{\partial x^{b_n}} \frac{\partial x^{a_1}}{\partial \overline{x}^{i_1}} \cdots \frac{\partial x^{a_m}}{\partial \overline{x}^{i_m}} \overline{\omega}^{i_1 \cdots i_m}_{j_1 \cdots j_n},\end{aligned} \tag{2.47}$$

where $\overline{\omega}^{i \cdots i}_{j \cdots j} = \overline{\tau}\, \overline{T}^{i \cdots i}_{j \cdots j}$; (2.48)

this transformation proves that ω is not a true tensor.

An important consequence of the definitions of tensor densities and tensor capacities is: <u>the product of a tensor density and a tensor capacity gives a true tensor.</u> This can easily be demonstrated. The extra factor in such a product of a tensor density and a tensor capacity [Eq. (2.44) and Eq. (2.47)] which might prevent the product from being a true tensor is

$$\gamma \tau = \Delta \Delta^{-1} \overline{\gamma}\, \overline{\tau} = \overline{\gamma}\, \overline{\tau} = \text{true scalar}, \tag{2.49}$$

from which the tensor character of the product follows.

19. <u>The volume element as a scalar capacity</u>

The exterior product of n infinitesimal vectors $\delta_1 x$, $\delta_2 x$ ....., $\delta_n x$ is given by

$$\delta v^{\alpha \dots \mu} = \left\| \delta_1 x^\alpha \, \delta_2 x^\beta \dots \delta_n x^\mu \right\| = \begin{vmatrix} \delta_1 x^\alpha & \delta_1 x^\beta & \dots & \delta_1 x^\mu \\ \delta_2 x^\alpha & \delta_2 x^\beta & \dots & \delta_2 x^\mu \\ \dots & \dots & \dots & \dots \\ \delta_n x^\alpha & \delta_n x^\beta & & \delta_n x^\mu \end{vmatrix} \tag{2.50}$$

$$\begin{aligned} &+\delta\tau \text{ - (even permutation)} \\ = \; &\;\; 0 \;\text{ - (two equal indices)} \\ &-\delta\tau \text{ - (odd permutation)} \; . \end{aligned}$$

This determinant yields an element δ of the scalar-capacity type. The element is the infinitesimal volume of the hyper-parallelopiped constructed from the infinitesimal vectors.

The transformation formula of δ is given by [Eq. (2.42)]

$$\delta\tau = \Delta\,\delta\bar{\tau} \; ; \tag{2.51}$$

this equation is equivalent to

$$dx^1 \, dx^2 \dots dx^n = \frac{\partial(x^1 \dots x^n)}{\partial(\bar{x}^1 \dots \bar{x}^n)} d\bar{x}^1 d\bar{x}^2 \dots d\bar{x}^n \tag{2.52}$$

where $\dfrac{\partial(x^1 \dots x^n)}{\partial(\bar{x}^1 \dots \bar{x}^n)}$ is the functional determinant. It should be noted that no measure of volume capacity has been introduced.

20. The surface element as an anti-symmetric tensor

An element of surface can be represented by a second order anti-symmetric tensor of the exterior product type. If two vectors $\delta_1 x$ and $\delta_2 x$ form the sides of a surface element, then the tensor whose components are

$$\delta S^{\alpha} = \begin{vmatrix} \delta_1 x^{\alpha} & \delta_1 x \\ \delta_2 x^{\alpha} & \delta_2 x \end{vmatrix} = \delta_1 x^{\alpha} \quad \delta_2 x \quad - \delta_1 x \, \delta_2 x^{\alpha} \qquad (2.53)$$

has a value equal to <u>twice</u> the area of the surface element. This surface element will be used in Stoke's theorem in a later section.

III. DIFFERENTIAL GEOMETRY

All of the formulas which have been presented in Chapter II have merely required the existence of reference coordinate systems in an n-dimensional manifold; they have as yet been given no geometrical interpretation or significance. The introduction of a few additional postulates of mensuration and representation will give geometrical significance to these formulas; this introduction furnishes the basis for a universal geometrical language or method of reasoning; thereby divers physical problems can be expressed in the same language. Analogous equations of various origins can be visualized and interpreted on a common basis -- the properties of the various types of curves, surfaces, and spaces being iso-morphic with some physical systems. The application of this universal language to physical problems results in a "geometrization" of physics.

In the application of geometrical methods to physical problems it is usually necessary to idealise nature by making certain simplifying assumptions. The extent of the assumptions is almost an inverse measure of the complexity of the geometry required to represent a given problem; therefore many assumptions are usually required for complex problems. Almost instinctively the physicist will make the necessary assumptions for the application of Euclidean methods to a given problem; often satisfactory results are obtained. There are some problems, however, which are not accurately expressible in such a simplified form. It then becomes necessary to adopt one of the more general, and usually more complex geometries for the representation of such problems. The succeeding sections of this presentation of tensor theory are chiefly concerned with the general non-Riemannian and Riemannian geometries which include most other geometries as special cases. Before taking up these geometries, however, it is desirable to elicit some conception of what is meant by space.

1. Kinds of spaces

One of the major problems of philosophy through the ages has been that of determining what space is. There seems to be no such thing as space, but only spaces, because there is associated with the notion of space so many different ideas (13). Psychologically, space is only a mode of sense perception. Such a space (or spaces) is a private space; it is non-isotropic, discontinuous, and of uncertain dimensionality. For the sake of human intercourse, however, emphasis is places on a public or physical space; physical space is a mental abstraction of private space. Physical space is homogeneous, isotropic, three-dimensional, and Euclidean in its geometry; it is the space used almost instinctively by the physicist. This space is based upon the operational notations of space-interval or "measuring-stick," and the equality of space intervals; it possesses those properties of rigid bodies which are independent of their material content. Then there are the geometrical spaces of the mathematician. Geometrical spaces are of great importance in physical theory.

2. Geometrical spaces

The physical space mentioned in the last section is dependent upon actual experiments which are subject to uncertainty. Nevertheless, it has been possible to build a theory in which the concepts (points, lines, planes, etc.) are abstractions from experimental knowledge; but certain postulates which appear operationally reasonable must be assumed. This theory is geometry; the space it defines is a geometrical space.

The mathematician distinguishes clearly between two types of geometrical spaces; namely, vector spaces; metric spaces (2, p. 17). All of the important geometries are representable in one or both of the above types of space.

In the first type of space the particular postulates necessary for the definition of a vector in terms of a linear equation in the unit vectors along the several axes are admitted. The vector space of n dimensions will contain n coordinate axes on each of which is defined a particular unity or measuring stick $\overline{e}_1$; then a vector $\overline{v}$ can be represented at a given point in the vector space in terms of its components v^1, $v^2 \ldots v^n$ by the equation

$$v = v^{\alpha} \quad e_{\alpha} \; . \tag{3.1}$$

The same postulates which permit the definition of a vector constitute the basis of affine geometry. Affine geometry is characterized by the operations in the theory of linear equations; it is a non-metrical geometry with a symmetric connection [Eq. (3.11)]. The space defined by affine geometry is a linear space.

In the space of affine and vector geometries no relationships between the unities along the various axes exist; therefore such important physical quantities as the magnitude of a vector and the angle between two vectors cannot be defined; a very limited application to physical theory results.

When a generalization of the Pythagorean theorem

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (3.2)$$

for oblique infinitesimal lengths and n dimensions is introduced, this postulate, in effect, introduces a ratio between the unit vectors along the different axes and makes the vector or affine space a metrical space. The geometry of this metric space is known as metric geometry; it may be either a Riemannian or a non-Riemannian geometry; it is an infinitesimal geometry. The "linear" geometry of Euclid, the "hyperbolic" geometry of Bolyai and Lobatchevsky, and the "elliptic" geometry of Riemann are special cases of metric geometry.

The quadratic form [Eq. (3.2)] which defines the metrical properties of the space need not be positive. Riemann himself pointed out that this quadratic form was to be regarded as a physical reality, since it reveals itself in centrifugal forces (25, p. 202), for example, as the origin of real effects on matter; hence metric geometry is sometimes called "Natural" geometry. This radical change in the outlook upon the metrical properties of space as pertaining to space itself, independently of the matter it contained, is the basic idea which Einstein has since developed. Einstein's general theory of relativity is based on a negative quadratic form; it is an attempt to bring gravitation and electro-magnetic phenomena to the account of geometry (6). Before the introduction of Weyl's symmetrical non-Riemannian affine geometry these two classes of phenomena stood separately. Weyl's work more closely approached a complete unification of gravitation and electromagnetic phenomena than had the work of Einstein.

Weyl and others recognized that for an infinitesimal geometry to be in agreement with nature it must be based upon the fundamental conception of infinitesimal parallel displacement, instead of the finite displacement of Riemannian geometry. The Riemannian geometry seems to retain, probably due to its accidental origin in the theory of surfaces, an element of finite geometry. The metric quadratic form permits comparison, with respect to length, not only of two vectors at the same point, but also vectors at different points. But, according to Weyl (25, p. 203), "a truly infinitesimal geometry must recognize only the principle of the transference of a length from one point to another infinitely near to the first." This statement forbids the assumption that the transference of length from one point to another at a finite distance is integrable, even as the transference of direction is non-integrable. A truly infinitesimal geometry, in Weyl's opinion (25, p. 203) comes into being, which, when applied to the world, unites gravitation and electromagnetic phenomena in a theory in which "all physical quantities have a meaning in world geometry." The structure of this geometry is given consideration on succeeding pages. Both Weyl's and Riemann's geometries are special cases of the non-Riemannian geometry. On account of their physical significance these metrical geometries are sometimes called "Natural" geometries (5 , p. 362).

3. Difficulties and precautions in the definition of tensor derivatives

All of the tensor quantities of the previous sections have significance in the non-metrical vector or affine space. These quantities are, in general, closely tied to a point of the space, although, in the case of Cartesian reference frames, no harm results from sliding the quantities to any point of the space. In affine space with Cartesian reference frames the operation of differentiation of tensor quantities assumes a very simple form; the various components are separately differentiated as in ordinary calculus. But when curvilinear axes are introduced, the transformation coefficients [Eq. (2.8)] are not constants but are functions of position; this fact makes necessary the introduction of an additional postulate in order to define tensor differentiation.

A partial derivative of a function is defined as the limit of the increment in value of the function between two neighboring points due to a small increase in one of the coordinates, while the others are maintained constant, divided by the increment of the coordinate as the increment is allowed to approach zero. Such also must define the procedure in the formation of tensor derivatives.

To define the partial derivative of a tensor, the values of the tensor at two neighboring points, P and P', must be compared. In general, the components of a tensor have different transformation coefficients at two different points; this means that the rule for tensor addition and subtraction is not valid for such a case; therefore the partial derivative formed in the usual way is not a tensor. It becomes necessary to introduce a postulate of comparison before a covariant derivative operator, which, when applied to a tensor, furnishes a derivative which is a true tensor, can be defined.

4. The postulate of parallel displacement and the affine connection

Weyl (25, p. 205) has shown that it is possible to introduce the covariant derivative operator without specifying a metric as was previously thought necessary; the choice of a metric, however, does permit simple definition of a covariant derivative.

In order to define a covariant derivative, following Brillouin (2, p. 76) who uses Weyl's method of approach, it is first necessary to transport a tensor defined at one point P to a neighboring point P'. Starting with a vector u (x^i) which is defined at P $(x^1, x^2, \ldots\, x^n)$, the condition for its parallel displacement to P' $(x^1 + \delta x^1, \ldots\, x^n + \delta x^n)$ will be defined. For some particular reference system the numeric equality of the components u^i of u at P and P' can be admitted as definition without loss of generality because this condition will no longer be realized in another reference system. This condition is assumed realized in the coordinate system $\bar{x}^i$; therefore

$$\bar{u}^j_D(\bar{x}^i + \delta \bar{x}^i) = \bar{u}^j(\bar{x}^i). \qquad (3.3)$$

The subscript D stands for a displaced evaluation. If both sides of this equation are transformed to any other coordinate system x^k, the result is that

$$u^i(x^k) = \frac{\partial x^i}{\partial \bar{x}^\alpha}(x^k)\, \bar{u}^\alpha, \qquad (3.4)$$

and

$$u_D^i(x^k + \delta x^k) = \frac{\partial x^i}{\partial \bar{x}^\alpha}(x^k + \delta x^k)\,\bar{u}^\alpha\ , \qquad (3.5)$$

where $\frac{\partial x^i}{\partial \bar{x}^j}(x^k)$, and $\frac{\partial x^i}{\partial \bar{x}^j}(x^k + \delta x^k)$ mean that these transformation coefficients are functions of x^k and $(x^k + \delta x^k)$, respectively. By utilizing Taylor's expansion theorem, $u_D^i(x^k + \delta x^k)$ can be expressed in terms of the value of $u^i(x^k)$ and the values of the partial derivatives of $u^i(x^k)$ at the point P. This operation yields, neglecting the higher terms,

$$u_D^i(x^k + \delta x^k) = u^i(x^k)\Big|_P + \frac{\partial u^i}{\partial x^k}\Big|_P \delta x^k. \qquad (3.6)$$

Since

$$u^i(x^k) = \frac{\partial x^i}{\partial \bar{x}^\alpha}(x^k)\,\bar{u}^\alpha, \qquad (3.7)$$

it follows that

$$\frac{\partial u^i(x^k)}{\partial x^k} = \frac{\partial^2 x^i}{\partial \bar{x}^l\,\partial \bar{x}^\alpha}\quad \frac{\partial \bar{x}^l}{\partial x^k}\,\bar{u}^\alpha(\bar{x}^i). \qquad (3.8)$$

The substitution of (3.8) in equation (3.6) yields

$$u_D^i(x^k + \delta x^k) = u^i(x^k) + \frac{\partial^2 x^i}{\partial \bar{x}^l\,\partial \bar{x}^\alpha}\,u^\alpha\,\frac{\partial \bar{x}^l}{\partial x^k}\,\delta x^k \qquad (3.9)$$

or

$$u_D^i(x^k + \delta x^k) = u^i(x^k) + \frac{\partial^2 x^i}{\partial \bar{x}^l\,\partial \bar{x}^\alpha}\,u^m\,\delta x^k\,\frac{\partial \bar{x}^\alpha}{\partial x^m}\,\frac{\partial \bar{x}^l}{\partial x^k} =$$

$$u^i(x^k) - L^i_{km}\,u^m\,\delta x^k, \qquad (3.10)$$

where

$$L^i_{km} = \frac{\partial^2 x^i}{\partial \bar{x}^l\,\partial \bar{x}^\alpha}\cdot\frac{\partial \bar{x}^\alpha}{\partial x^m}\,\frac{\partial \bar{x}^l}{\partial x^k}\,. \qquad (3.11)$$

L^{i}_{km} can be shown to be a non-tensor quantity; it is called the affine connection. It is assumed here that the order of differentiation is immaterial; that is, that the symmetry

$$L^{i}_{km} = L^{i}_{mk} = \Gamma^{i}_{km} \tag{3.12}$$

where Γ^{i}_{km} is the symmetric part of L^{i}_{km}, exists. This assumption renders a great simplification in many formulas, but it is not necessary. Equations (3.11) give the conditions of parallel displacement of a vector u^{i}; that is, they give the value which the vector u evaluated at P assumes when it is moved parallel to itself to P'.

The very particular coordinates in which the components of the vector u are equal at the two points are called geodesic coordinates. There is an infinity of geodesic systems for infinitesimal displacements; in fact, any arbitrary linear transformation of coordinates which involves constant coefficients of transformation maintains the equality of the components. These various coordinates may have different curvatures; but they are still geodesic coordinates for sufficiently small displacements.

5. Covariant derivative of a vector

It is now possible to define a covariant derivative which yields a tensor of lower order. The true increase Du^{i} in a component of a vector u between two nearby points P and P', is equal to the value of the vector at the end point P' minus the value of the vector at the initial point P transported to the end point; that is, the true increase

$$Du^i = u^i(x^k + \delta x^k) - u^i(x^k) + \Gamma^i_{mk} u^m \delta x^k. \tag{3.13}$$

If this equation is divided by the true variation δx^k, the covariant derivative is obtained.

$$\frac{Du^i}{Dx^k} = U^i_{,k} = \frac{\partial u^i}{\partial x^k} + \Gamma^i_{mk} u^m \tag{3.14}$$

The covariant derivative of a covariant vector is formed in a similar way. The main difference in the results is that instead of a mixed tensor of the second order a covariant tensor of the second order is obtained; also different affine connections are obtained. The covariant derivative of the covariant vector v is given by

$$\frac{Dv_i}{Dx^k} = v_{i,k} = \frac{\partial v_i}{\partial x^k} + B^m_{ik} v_m . \tag{3.15}$$

The affine connections for the contravariant and covariant cases can be related easily by forming a contracted product of the two vectors and by using the relation that the covariant derivative of the scalar thus formed is the same as the partial derivative of the scalar; that is,

$$\frac{D}{Dx^k} (u^i v_i) = \left(\frac{\partial u^i}{\partial x^k} + \Gamma^i_{mk} u^m\right) v_i + u^i \left(\frac{\partial v_i}{\partial x^k} + B^m_{ik} v_m\right), \tag{3.16}$$

and

$$\frac{D}{Dx^k}(u^i v_i) = \frac{\partial}{\partial x^k} (u^i v_i) = \frac{\partial u^i}{\partial x^k} v_i + u^i \frac{\partial v_i}{\partial x^k} . \qquad (3.17$$

From equations (3.16 and (3.17) the result

$$\left(\frac{\partial u^i}{\partial x^k} + \Gamma^i_{mk}\, u^m\right) v_i + u^i \left(\frac{\partial v_i}{\partial x^k} + B^m_{ik}\, v_m\right) = \frac{\partial u^i}{\partial x^k}\, v_i + u^i\, \frac{\partial v_i}{\partial x^k}\,. \qquad (3.18)$$

is obtained. From equation (3.18 it follows that

$$-\Gamma^i_{mk}\, u^m\, v_i = B^i_{mk}\, u^m\, v_i. \qquad (3.19)$$

or

$$\Gamma^i_{mk} = -B^i_{mk}; \qquad (3.20)$$

where the affine connection for covariant differentiation is the negative of that for contravariant differentiation.

The covariant derivatives obey the same rules of products, sums, and repetitions as do ordinary derivatives; therefore these rules will not be repeated here.

6. Covariant derivatives of tensors

If $(u^i\, v_j)$ is used in equation (3.16) instead of $u^i\, v_i$, then the equation

$$\begin{aligned}\frac{D(u^i\, v_j)}{Dx^k} &= \left(\frac{\partial u^i}{\partial x^k} + \Gamma^i_{mk}\, u^m\right) v_j + \left(\frac{\partial v_j}{\partial x^k} - \Gamma^j_{mk} v_j\right) u^i \\ &= \frac{\partial (u^i\, v_j)}{\partial x^k} + \Gamma^i_{mk}\, u^m\, v_j - \Gamma^j_{mk}\, u^i\, v_j\end{aligned} \qquad (3.21)$$

is obtained. From the rule of formation of tensors Eq. (2.25) it is clear that $(u^i\, v_j)$ is a mixed tensor of the second order; therefore

$$\frac{D(t^i_j)}{Dx^k} = \frac{\partial t^m_j}{\partial x^k} + \Gamma^i_{mk}\, t^m_j - \Gamma^m_{jk}\, t^i_m \qquad (3.22)$$

This method of forming rules of tensor differentiation can be generalized for the product of P vectors or for a general mixed tensor of order P. The rule for formation of the covariant derivative of the tensor T with components $T^{ab\ldots c}_{ij\ldots k}$ can be expressed thus:

$$\frac{D(T^{ab\ldots c}_{ij\ldots k})}{Dx^l} = \frac{\partial(T^{ab\ldots c}_{ij\ldots k})}{\partial x^l} + \Gamma^{a}_{ml} T^{mb\ldots c}_{ij\ldots k}$$

$$+ \Gamma^{b}_{ml} T^{am\ldots c}_{ij\ldots k} + \ldots + \Gamma^{c}_{ml} T^{ab\ldots m}_{ij\ldots k}$$

$$- \Gamma^{m}_{il} T^{ab\ldots c}_{mj\ldots k} - \Gamma^{m}_{jl} T^{ab\ldots c}_{im\ldots k} - \ldots$$

$$- \Gamma^{m}_{kl} T^{ab\ldots c}_{ij\ldots m}. \tag{3.23}$$

7. <u>Covariant derivatives of pseudo-tensors</u>

In order to determine the formula for the covariant derivative of a scalar density or capacity it is convenient to start with either a product of a scalar density and a contravariant vector, or a scalar capacity and a covariant vector. The partial derivative with respect to x of the vector density

$$A^k = Au^k \tag{3.24}$$

is first formed. This derivative is given by

$$\frac{\partial A^k}{\partial x^k} = \frac{\partial (Au^k)}{\partial x^k} = \left(u^k \frac{\partial A}{\partial x^k} + A\frac{\partial u^k}{\partial x^k}\right). \tag{3.25}$$

If it is assumed that the true change Du^k in u^k for a parallel displacement is zero, equation (3.14) gives

$$\frac{\partial u^k}{\partial x^k} = -\Gamma^k_{mk}\, u^m. \tag{3.26}$$

The substitution of this result in equation (3.25) produces

$$\frac{\partial A^k}{\partial x^k} = u^k \frac{\partial A}{\partial x^k} - A\Gamma^k_{mk}\, u^m \,. \tag{3.27}$$

Since it is permissible to interchange the dummy indices, this equation is equivalent to

$$\frac{\partial A^k}{\partial x^k} = u^k \left(\frac{\partial A}{\partial x^k} - A\Gamma^m_{km} \right). \tag{3.28}$$

This partial derivative is a scalar density because there is a contraction of the indices. The components u^k represent a contravariant vector; therefore in order that both sides of the equations be scalar densities, the term in the parentheses must be a covariant vector density. This result yields the following definition for the covariant derivative of a scalar density:

$$\frac{DA}{Dx^k} = \frac{\partial A}{\partial x^k} - A\Gamma^m_{km} \,. \tag{3.29}$$

Similarly, for a scalar capacity

$$\frac{D}{Dx^k} = \frac{\partial}{\partial x^k} + \Gamma^m_{km} \,. \tag{3.30}$$

After the covariant derivatives of scalar densities and of scalar capacities have been defined the covariant derivatives of tensor densities and of tensor capacities easily follow by the customary product rule of differentiation and equations (3.28), (3.29), and (3.30). The covariant derivative of a tensor density T is given by

$$
\begin{aligned}
\frac{D}{Dx^l}\, T^{ab\ldots c}_{ij\ldots k} &= \frac{D}{Dx^l}\, A T^{ab\ldots c}_{ij\ldots k} \\
&= \frac{\partial}{\partial x^l}\, A T^{ab\ldots c}_{ij\ldots k} + A\Big[\Gamma^{a}_{ml}\, T^{mb\ldots c}_{ij\ldots k} + \ldots\ldots \\
&\quad \Gamma^{c}_{ml}\, T^{ab\ldots m}_{ij\ldots k}\Big] - A\Big[\Gamma^{m}_{il}\, T^{ab\ldots c}_{ij\ldots k} + \ldots\ldots \\
&\quad + \Gamma^{m}_{il}\, T^{ab\ldots c}_{ij\ldots m}\Big] - A\Big[\Gamma^{m}_{lm}\, T^{ab\ldots c}_{ij\ldots k} \quad \ldots\ldots \\
&= \frac{\partial}{\partial x^k}\, \overset{D}{T}{}^{ab\ldots c}_{ij\ldots k} + \Big[\Gamma^{a}_{ml}\, T^{mb\ldots c}_{ij\ldots k} + \ldots\ldots \\
&\quad \Gamma^{c}_{ml}\, \overset{D}{T}{}^{ab\ldots m}_{ij\ldots k}\Big] - \Big[\Gamma^{m}_{il}\, T^{ab\ldots c}_{mj\ldots k} + \ldots\ldots \\
&\quad + \Gamma^{m}_{kl}\, \overset{D}{T}{}^{ab\ldots c}_{ij\ldots m} + \Gamma^{m}_{ml}\, T^{ab\ldots c}_{ij\ldots k}\Big], \qquad (3.31)
\end{aligned}
$$

where the D over the T is used to indicate the tensor-density character of the component.

A similar result is obtained for a tensor capacity. The only difference between the derivative of a tensor density and tensor capacity is that the last term $+\Gamma^{m}_{ml}\, \overset{D}{T}{}^{ab\ldots c}_{ij\ldots k}$ of equation (3.31) is replaced by $-\Gamma^{m}_{ml}\, \overset{C}{T}{}^{ab\ldots c}_{ij\ldots k}$, where the C over the T represents tensor capacity character of the component.

8. Absolute derivative of a vector

Sometimes the coordinates of a mobile point in a space of r dimensions is expressed as a function of some parameter t. If a contravariant vector u is carried along with the point, it may be desirable to determine the true change Du^i of the components of u when u is subjected to a parallel displacement due to a change in t. It follows from equation (3.13) that

$$\frac{Du^i}{Dt} = \frac{du^i}{dt} + \Gamma^{\,i}_{mk}\, u^m \frac{dx^k}{dt}\,. \tag{3.32}$$

These expressions represent the components of the absolute or intrinsic derivative of the vector u.

9. Geodesic lines

The vector u of equation (3.32) might be given as the velocity of the mobile point; that is, it might be represented by the components

$$u^i = \frac{dx^i}{dt}. \tag{3.33}$$

In this case the absolute or intrinsic derivative would give the true absolute acceleration of the mobile point; that is,

$$\frac{Du^i}{Dt} = \frac{d^2x^i}{dt^2} + \Gamma^{\,i}_{mk} \frac{dx^m}{dt} \frac{dx^k}{dt} \tag{3.34}$$

would be the true acceleration. For the motion to occur at constant velocity, $\frac{D}{Dt}\left(\frac{dx^i}{dt}\right)$ must equal zero; then

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{mk}\frac{dx^m}{dt}\frac{dx^k}{dt} = 0. \qquad (3.35)$$

This expression defines a small segment of a straight line; the gradual junction of these small segments would give a geodesic line.

10. Finite displacements of scalar quantities

The conditions for infinitesimal parallel transport of tensors have been given. Can these transports be extended to finite distances? This question can be answered quite easily for scalar quantities by starting with the values of a scalar φ at a point P and at a nearby point P', following Brillouin (2, p. 86), by specifying what factors determine the change in the value of φ for a parallel displacement from P to P', and afterwards by extending the displacement to finite intervals, a criterion can be obtained. If δx^k are the components of the vector $\delta x = PP'$, the variation of φ will be proportional to these components; that is,

$$\delta\varphi = f_k\ \delta x^k, \qquad (3.36)$$

where f_k is a component of a covariant vector which serves as a proportionality factor. If Q is a point at a finite distance, it can be approached along some path PP'AQ. The value of φ at Q must be equal to the value of φ at P plus the change in φ; this value can be expressed by allowing δx in equation (3.36) to decrease to dx and by integrating the resulting expression along the path PP'AQ; the result is that

$$\varphi_Q = \varphi_P + \int_{PP'AQ} f_k\, dx^k. \qquad (3.37)$$

If equation (3.37) is an integrable relation, then it must be true for any other path PBQ. For the path PBQ

$$\varphi_Q' = \varphi_P + \int_{PBQ} f_k \, dx^k. \tag{3.38}$$

It is clear that the line integral

$$\varphi_Q - \varphi_Q' = \int_{PP'AQBP} f_k \, dx^k \tag{3.39}$$

must vanish for integrability. It can be shown that for this integral to be zero that f_k must be the gradient of the scalar function φ. For this demonstration it is desirable to use Stoke's Theorem.

11. Stoke's theorem

Stoke's theorem is a very useful theorem by means of which a line integral around a closed contour of a surface is expressed in terms of an integral over the surface. A covariant vector function with components f_k and its first derivatives are assumed continuous and single valued over a surface S with a contour C. The line integral of this function around C is then given by:

$$I = \oint_c f_k \, dx^k. \tag{3.40}$$

The surface S which is bounded by c can be made a part of a two-dimensional coordinate surface by the proper choice of the coordinate system. If $\bar{x}^i$ is the new coordinate system and $\bar{x}^1$ and $\bar{x}^2$ are the variable coordinates on the surface S, then the remaining coordinates $\bar{x}^3, \ldots, \bar{x}^n$ are constants.

The surface S can now be divided into small elements ABCDA formed by the coordinate lines. At $A(\bar{x}^i)$, the vector has the components $\bar{f}_1 \ldots \bar{f}_n$ in the new coordinate system. The line integral of f around ABCDA (traversing ABCDA and C in the same direction) is given by the sum of the four components

$$f_1 \;\delta\bar{x}^1 \qquad\qquad \text{along AB;}$$

$$(\bar{f}_2 + \frac{\partial f_2}{\partial \bar{x}^1}\delta\,\bar{x}^1)\;\delta\bar{x}^2 \qquad\qquad \text{along BC;}$$

$$-\,(\bar{f}_1 + \frac{\partial \bar{f}_1}{\partial \bar{x}^2}\;\delta\bar{x}^2)\;\delta\bar{x}^1 \qquad\qquad \text{along CD;}$$

$$-\;\bar{f}_2\;\;\delta\bar{x}^2 \qquad\qquad \text{along DA;}$$

The sum is given by

$$\frac{\partial \bar{f}_2}{\partial \bar{x}^1}\delta\bar{x}^1\delta\bar{x}^2 - \frac{\partial \bar{f}_1}{\partial \bar{x}^2}\;\;\delta\bar{x}^1\delta\bar{x}^2 \qquad\qquad \text{along ABCDA;}$$

therefore it follows that

$$\delta\,I = \frac{\partial \bar{f}_2}{\partial \bar{x}^1} - \frac{\partial \bar{f}_1}{\partial \bar{x}^2}\;\;\delta\bar{x}^1\;\;\delta\,\bar{x}^2$$

$$= \;r_{12}\;\;\delta\bar{x}^1\;\;\delta\,\bar{x}^2\;. \tag{3.41}$$

where

$$r_{12} = \frac{\partial \bar{f}_2}{\partial \bar{x}^1} - \frac{\partial \bar{f}_1}{\partial \bar{x}^2}\;. \tag{3.42}$$

But

$$\bar{r}_{12} = \frac{\partial x^i}{\partial \bar{x}^1}\cdot\frac{\partial x^i}{\partial \bar{x}^2}\cdot r_{ij}\;. \tag{3.43}$$

therefore

$$\delta I = r_{ij} \frac{\partial x^i}{\partial \bar{x}^1} \cdot \frac{\partial x^j}{\partial \bar{x}^2} \quad \delta \bar{x}^1 \; \delta \bar{x}^2 ; \tag{3.44}$$

this equation is equivalent to

$$\delta I = \frac{1}{2} r_{ij} \, ds^{ij} . \tag{3.45}$$

The factor $\frac{1}{2}$ in equation (3.45) is necessary because $ds^{ij} = \delta x^i \; \delta x^j - \delta x^j \; \delta x^i$ [Eq. (2.53)] ; that is, the area is represented twice by the presence of the anti-symmetric tensor ds^{ij}. Stoke's theorem is now obtained by integrating equation (3.45) over the entire surface S; this operation yields

$$I = \frac{1}{2} \iint_s \quad r_{ij} \, ds^{ij} , \tag{3.46}$$

or

$$\oint_c f_k \, dx^k = \frac{1}{2} \iint_S r_{ij} \, ds^{ij} , \tag{3.47}$$

where

$$r_{ij} = \frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} , \text{ the curl of } f. \tag{3.48}$$

Equation (3.47) is known as Stoke's theorem.

12. Integrability conditions for parallel displacement of a scalar

By means of Stoke's theorem, equation (3.39) can be expressed in the form

$$\varphi_Q - \varphi_{Q'} = \frac{1}{2} \qquad r_{ij} \, ds^{ij} . \tag{3.49}$$

In order that $\varphi_Q - \varphi_{Q'}$ vanish, r_{ij} must vanish. In order that

$$r_{ij} = \frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} = 0 \qquad (3.50)$$

be satisfied, f_i and f_j must be the components of the gradient of some scalar function φ ; then

$$\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - \frac{\partial^2 \varphi}{\partial x^j \partial x^i} = 0. \qquad (3.51)$$

If r_{ij} is such a curl of the gradient of a scalar function, then the line integral $\int_c f_k \, dx^k$ is zero, and the scalar function φ has a unique value at each point; but if the line integral $\int_c f_k \, dx^k$ is not zero, the values of φ can be compared only in the infinitely small region surrounding a particular point.

13. Finite displacement of vector quantities

If the value of a vector u at a point P is taken and displaced parallel to itself to a nearby point P', the condition of parallelism is, by equation (3.12), that

$$Du^i = du^i + \Gamma^i_{mk} u^m dx^k = 0. \qquad (3.52)$$

This condition yields the variation

$$\begin{aligned} du^i &= - \Gamma^i_{mk} u^m dx^k \\ &= f^i_k \, dx^k, \end{aligned} \qquad (3.53)$$

where

$$f^i_k = - \Gamma^i_{mk} u^m. \qquad (3.54)$$

The line integral of du^i around a finite closed path C, bounding a surface S can now be formed and transformed to a surface integral by means of Stoke's theorem [Eq. (3.47)]. These operations yield the change

$$\delta u^i = \oint f^i_k \, dx^k = \frac{1}{2} \iint_S r^i_{kh} \, ds^{kh}, \tag{3.55}$$

where

$$r^i_{kh} = \frac{\partial f^i_h}{\partial x^k} - \frac{\partial f^i_k}{\partial x^h} . \tag{3.56}$$

For integrable transport, r^i_{kh} must be zero; if r^i_{kh} is not zero, the integration is impossible. In terms of f^i_k, as defined by equation (3.54), r^i_{kh} is given by

$$\begin{aligned} r^i_{kh} &= \frac{\partial}{\partial x^k}\left(-\Gamma^i_{mh} u^m\right) - \frac{\partial}{\partial x^h}\left(-\Gamma^i_{mk} u^m\right) \\ &= u^m\left(\frac{\partial \Gamma^i_{mk}}{\partial x^h} - \frac{\partial \Gamma^i_{mh}}{\partial x^k}\right) + \Gamma^i_{mk} \frac{\partial u^m}{\partial x^h} - \Gamma^i_{mh} \frac{\partial u^m}{\partial x^k} . \end{aligned} \tag{3.57}$$

By equations (3.53) and (3.54)

$$\frac{\partial u^m}{\partial x^h} = -\Gamma^m_{lh} u^l ;$$

and

$$\frac{\partial u^m}{\partial x^k} = -\Gamma^m_{lk} u^l . \tag{3.58}$$

The substitution of equation (3.58) into (3.57) gives

$$
\begin{aligned}
r^{i}_{kh} &= u^{m}\left(\frac{\partial \Gamma^{i}_{mk}}{\partial x^{h}} - \frac{\partial \Gamma^{i}_{mh}}{\partial x^{k}}\right) - \Gamma^{i}_{lk}\ \Gamma^{l}_{mh}\ u^{m} + \Gamma^{i}_{lh}\ \Gamma^{l}_{mk} u^{m} \\
&= u^{m}\left(\frac{\partial \Gamma^{i}_{mk}}{\partial x^{h}} - \frac{\partial \Gamma^{i}_{mh}}{\partial x^{k}} - \Gamma^{i}_{lk}\ \Gamma^{l}_{mh} + \Gamma^{i}_{lh}\ \Gamma^{l}_{mk}\right) \\
&= -\, u^{m}\ R^{i}_{m,kh}, \qquad (3.59)
\end{aligned}
$$

where

$$
R^{i}_{m,kh} = -\left(\frac{\partial \Gamma^{i}_{mk}}{\partial x^{h}} - \frac{\partial \Gamma^{i}_{mh}}{\partial x^{k}} - \Gamma^{i}_{lk}\ \Gamma^{l}_{mh} + \Gamma^{i}_{lh}\ \Gamma^{l}_{mh}\right) \qquad (3.60)
$$

is a component of a tensor.

The displacement of the vectors will be integrable if the expressions R are all zero. The fourth order tensor R is frequently called the <u>curvature tensor</u>. It can be used as a criterion for the identification of the different types of spaces; Euclidian space, for example, has zero curvature.

Just as a contravariant vector was subjected to a parallel transport to a finite distance, so also can a covariant vector. The condition of parallelism for a covariant vector is given by

$$
Dv_{j} = dv_{j} - \Gamma^{m}_{jk}\ v_{m}\ \ x^{k} = 0, \qquad (3.61)
$$

from which

$$
\begin{aligned}
dv_{j} &= \Gamma^{m}_{jk}\ v_{m}\ \ x^{k} \qquad (3.62) \\
&= f_{jk}\ dx^{k},
\end{aligned}
$$

where

$$
f_{jk} = \Gamma^{m}_{jk}\ v_{m}. \qquad (3.63)
$$

The change in v_i due to a parallel displacement around the contour c is given by

$$v_j = \iint_s r_{j,kh}\, ds^{kh}, \tag{3.64}$$

where

$$r_{j,kh} = \frac{\partial f_{jh}}{\partial x^k} - \frac{\partial f_{jk}}{\partial x^h} \,. \tag{3.65}$$

By the substitution of equation (3.62) into equation (3.65) , and equation (3.65) into equation (3.64), the following result is obtained:

$$v_j = \frac{1}{2} \iint_s R^i_{j,kh}\, ds^{kh} \tag{3.66}$$

where

$$R^i_{j,kh} = \frac{\partial}{\partial x^k} \Gamma^i_{jh} - \frac{\partial}{\partial x^h} \Gamma^i_{jk} + \Gamma^i_{lk} \Gamma^l_{jh}$$

$$- \Gamma^i_{lh} \Gamma^l_{jk} \,. \tag{3.67}$$

As before, the curvature tensor must be zero for integrability; otherwise, the value of the line integral of a covariant vector is zero. This statement is equivalent to saying that the order of forming the successive covariant derivatives of a vector is important; the result obtained from two successive differentiations depends upon the order of differentiation.

The method of attack used here can be applied to any tensor. The general rule for a tensor of a given order is obtained by expressing the tensor in a vector product form [Eq. (2.25)] and by using the product rule for differentials.

The Ricci-Einstein contracted curvature tensor can be easily formed from $R^i_{j,kh}$, by contracting the i and k indices to give

$$R_{j,h} = R^i_{j,ih} = \frac{\partial \Gamma^i_{jh}}{\partial x^i} - \frac{\partial \Gamma^i_{ji}}{\partial x^h} + \Gamma^i_{li}\,\Gamma^l_{jh} - \Gamma^i_{lh}\,\Gamma^l_{ji} \qquad (3.68)$$

This tensor is used extensively in relativity theory.

IV. METRIC GEOMETRY

The fundamental properties of tensors as they are related to an amorphous space have been sufficiently elicited. It now becomes desirable to consider them in relation to the fundamental quadratic form

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{4.1}$$

which converts the amorphous space into a metrical or Remannian manifold.

From the non-metrical point of view tensors or vectors are defined at points in the manifold, but, as an example, a vector at a specified point in the manifold cannot be compared in magnitude and direction with some different vector defined at the same point. Before such quantities as magnitudes (length) of vectors and direction of vectors, and the angle between two vectors can be specified, it is necessary to introduce some relationship between the various unities along the different axes at each point of the space. This relationship is provided by the fundamental quadratic form [Eq. (4.1)]. This quadratic form defines at each point of the n manifold and n - 1 dimensional hypersurface surrounding the point. All vectors originating at the central point and ending on the hypersurface are now said to be equal in length, or in absolute value. In the quadratic form ds is the length or magnitude of the vector whose components are dx^i.

By applying the quotient law to the fundamental quadratic form, it follows that g_{ij} is a component of a tensor of the second order; this tensor has, in general, n^2 independent components. It so happens that

$$g_{ij} = g_{ji}, \tag{4.2}$$

therefore only $\frac{n(n+1)}{2}$ of n^2 components are independent; that is, g_{ij} is a symmetric tensor of the second order. It will later be clear that this symmetry just means that for the case of rectilinear axes, for example, that the cosine of the angle between two components dx^i and dx^j does not depend upon whether the ratio of the projection of dx^i on dx^j to dx^j, or the ratio of the projection of dx^j on dx^i to dx^i is taken as the definition of the cosine of the angle between them.

1. Fundamental properties of the metric tensor

The components of the fundamental metric tensor can be arranged in the tabular form

$$g_{ij} = \begin{array}{c|cccc|c} & 1 & 2 & \cdots & n & \\ \hline i\downarrow & g_{11} & g_{12} & \cdots & g_{1n} & 1 \\ & g_{21} & g_{22} & \cdots & g_{2n} & 2 \\ & \cdots & \cdots & \cdots & \cdots & \\ & \cdots & \cdots & \cdots & \cdots & \\ & g_{n1} & g_{n2} & \cdots & g_{nn} & n \\ \hline & & j\rightarrow & & & \end{array} \tag{4.3}$$

The determinant of these components is given by

$$g = |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{vmatrix} \tag{4.4}$$

This determinant can be expanded by minors with respect to a row or a column to give

$$g = \sum_j g_{ij} G_{ij} \text{ (expansion with respect to row i),} \quad (4.5)$$

$$g = \sum_i g_{ij} G_{ij} \text{ (expansion with respect to column j),} \quad (4.6)$$

where G_{ij} is the cofactor of the ith row and jth column.

From determinant theory, it is known that

$$\sum_i g_{ij} G_{ik} = 0, \text{ and that } \sum_i g_{ji} G_{ki} = 0, \ j \neq k \quad (4.7)$$

If

$$g^{ij} = \frac{G_{ij}}{g}, \quad (4.8)$$

then the above relations can be condensed in the forms

$$g_{ij} g^{ik} = \delta^k_j = \begin{matrix} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{matrix}$$

$$g_{ji} g^{ki} = \delta^k_j = \begin{matrix} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{matrix} \quad (4.9)$$

The tensor g^{ij} is called the fundamental contravariant tensor; it will prove very useful in raising indices in later sections. The determinant of g^{ij} is equal to the inverse of the determinant of g_{ij}; that is,

$$g' = \left| g^{ij} \right| = g^{-1} \quad (4.10)$$

2. <u>The magnitude of a vector; the scalar product</u>

The fundamental quadratic form $ds^2 = g_{ij}\, dx^i\, dx^j$ gives the magnitude ds of the vector whose components are dx^i. Associated with each point of a space is a reference coordinate system along the axes of which the components of any vector u may be represented. Just as for the infinitesimal vector dx, the magnitude of any contravariant vector u can be represented by

$$|u|^2 = g_{ij}\, u^i\, u^j. \tag{4.11}$$

If the vector happens to be a covariant vector v_i, its magnitude is, by definition,

$$|v|^2 = g^{ij}\, v_i\, v_j. \tag{4.12}$$

It now becomes desirable to associate with each covariant vector a contravariant vector, and with each contravariant vector a covariant vector. The associate contravariant vector of a covariant vector has the components

$$v^i = g^{ki}\, v_k \quad (k = 1 \ldots n), \tag{4.13}$$

from which the original covariant form is easily obtained by multiplying by g_{ji} and summing with respect to i; the result is that the following relations exist:

$$\begin{aligned} g_{ji}\, v^i &= g_{ji}\, g^{\alpha i}\, v_\alpha \\ v_j &= \delta^\alpha_j\, v_\alpha \\ v_j &= v_j \end{aligned} \tag{4.14}$$

In this same way, covariant components or contravariant components can be associated with any tensor. By using the associate components of the vector, equations (4.11) and (4.12) can be simplified to

$$|u|^2 = u_i\, u^i \tag{4.15}$$

and

$$|v|^2 = v^i\, v_i \tag{4.16}$$

These last two equations suggest the scalar product of vector analysis. If two vectors u and v are specified, then their <u>scalar product</u> is defined by

$$\begin{aligned}(u.v) &= g_{ij}\, u^i\, v^j = g^{ij}\, u_i\, v_j \\ &= u_j\, v^i = u^j\, v_i\end{aligned} \tag{4.17}$$

Equation (4.17), in turn, suggests the cosine of the angle between the vectors u and v. From vector analysis, the scalar product is defined by

$$(u.v) = \left|u\right| . \left|v\right| \cos\,(\widehat{u\,v}) \tag{4.18}$$

In terms of relations given in this treatment, equation (4.18)becomes

$$(u.v) = \sqrt{g_{ij}\, u^i\, u^j}\ .\ \sqrt{g_{ij}\, v^i\, v^j}\ .\ \cos\,(\,\widehat{u\,v}), \tag{4.19}$$

from which

$$\begin{aligned}\cos\,(u\ v) &= \frac{(u\ .\ v)}{\sqrt{(g_{ij}\, u^i\, u^j)(g_{ij}\, v^i\, v^j)}} = \\ &\frac{g_{ij}\, u^i\, v^j}{\sqrt{(g_{ij}\, u^i\, u^j)(g_{ij}\, v^i\, v^j)}}\end{aligned} \tag{4.20}$$

is obtained.

Similar relations easily follow for the two sets of associate components u_i and v_i.

3. Transformation of the determinant g; the measurable volume element

g_{ij} is a component of a second order tensor; therefore its formula of transformation is given by

$$\bar{g}_{ij} = \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial x^m}{\partial \bar{x}^j} g_{lm}. \tag{4.21}$$

The determinant of $\bar{g}_{ij}$ can be formed and expressed in terms of the determinant of g_{ij} and the determinants of the transformation matrices. By a rule from determinant theory,

$$\left|\bar{g}_{ij}\right| = \left|\frac{\partial x^l}{\partial \bar{x}^i} \cdot \frac{\partial x^m}{\partial \bar{x}^j} g_{lm}\right| = \left|\frac{\partial x^l}{\partial \bar{x}^i}\right| \cdot \left|\frac{\partial x^m}{\partial \bar{x}^j}\right| \cdot \left|g_{lm}\right| \tag{4.22}$$

or

$$\bar{g} = \Delta^2 g,$$

where

$$\Delta = \left|\frac{\partial x^l}{\partial \bar{x}^i}\right| = \left|\frac{\partial x^m}{\partial \bar{x}^j}\right|. \tag{4.23}$$

The resemblance of this transformation formula to that of a scalar density is at once apparent. By taking the square root of the magnitude of both sides of equation (4.22) a typical scalar density

$$\sqrt{\bar{g}} = \sqrt{g} \tag{4.24}$$

is formed.

It has been demonstrated previously [Eq. (2.49)] that the product of a tensor capacity and a tensor density is a true tensor. This principle becomes useful in defining an element of measurable volume or volume capacity. By choosing the scalar density $\sqrt{g}$ of the metric determinant as the scalar density which is multiplied by the volume element $d\tau$, a scalar capacity, a true scalar (tensor of zero order)

$$dV = \sqrt{g}\ d\tau = \sqrt{g}\ dx^1\, dx^2 \ldots\ldots dx^n \qquad (4.25)$$

is formed. The scalar dV represents an element of volume; the scalar density $\sqrt{g}$ can be thought of as a density of volumetric content. A scalar density which included the density of the matter or charge in the volume element might also have been used in equation (4.25). In this case the mass or charge in the elemental volume is given by

$$dm = \rho\, d\tau, \qquad (4.26)$$

where ρ is a scalar density of matter or charge.

4. Reduction of g_{ij} to the diagonal form

In general, the components of the fundamental tensor g_{ij} are functions of position; if their values at a given point are taken, the tabular form of representing the components of g_{ij}.

$$g_{ij} = \begin{array}{c|c|c} & 1 \quad 2 \quad \ldots\ldots \quad n & \\ i\downarrow & \begin{array}{cccc} g_{11} & g_{12} & \ldots\ldots & g_{1n} \\ g_{21} & g_{22} & \ldots\ldots & g_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ g_{n1} & g_{n2} & \ldots\ldots & g \end{array} & \begin{array}{c} 1 \\ 2 \\ \\ n \end{array} \\ & j \longrightarrow & \end{array} \qquad (4.27)$$

can be reduced to diagonal form. The diagonal form of the g_{ij} components represents geometrically an Euclidean space locally associated with the given point. This principle is often used in relativity theory.

To obtain the diagonal form of equation (4.27) a new axis $\bar{x}^1$ for which the magnitude ds_1^2 of its differential may be either positive or negative is chosen in order to derive some guiding principles. The unity $\bar{e}_1$ is defined along the axis $\bar{x}^1$ as ± 1 depending upon whether ds_1^2 is positive or negative. The projection of $d\bar{x}^1$ along any axis x^i is

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^1} d\bar{x}^1. \tag{4.28}$$

In terms of the dx^i, the magnitude of dx^1 is, by equation (4.1),

$$ds_1^2 = g_{ij} \cdot \frac{\partial x^i}{\partial \bar{x}^1} \cdot \frac{\partial x^j}{\partial \bar{x}^1} (d\bar{x}^1)^2.$$

But equation (4.29) is equivalent to

$$ds_1^2 = \bar{g}_{11} (d\bar{x}^1)^2; \tag{4.29}$$

therefore, according to the definition of $\bar{e}_1$,

$$g_{11} = \pm 1. \tag{4.30}$$

The unities along the old axis x^i are related to the defined $\bar{e}^1$ by

$$\bar{e}_1 = \frac{\partial x^i}{\partial \bar{x}^1} e_i. \tag{4.31}$$

Equations (4.30) and (4.31) provide the desired principles.

A vector $\delta\vec{x}$ is projected on the axis $\bar{x}^1$ to give $\delta\bar{x}^1$; the remaining portion $\delta'\vec{x}$ is defined in an $(n - 1)$ dimensional space, forming a hypersurface in an n-dimensional space, orthogonal to $\delta\bar{x}^1$; that is, $\delta\vec{x}$ is divided to give

$$\delta\vec{x} = \delta\bar{x}^1 + \delta'\vec{x} \tag{4.32}$$

where $\delta' x$ lies in the hypersurface defined by

$$\sum_{ji} g_{ij} \quad \delta' x^i \frac{\partial x^j}{\partial \bar{x}^1} \delta \bar{x}^1 = 0 \tag{4.33}$$

In terms of δx, as expressed in equation (4.32) the magnitude of δx is given by

$$\delta s^2 = g_{ij} \quad \frac{\partial x^i}{\partial \bar{x}^1} \delta \bar{x}^1 + \delta x^i \quad \frac{\partial x^j}{\partial \bar{x}^1} \delta \bar{x}^1 + \delta x^j$$

$$= \bar{g}_{11} \; (\; \bar{x}^1)^2 + g_{ij} \quad \delta' x^i \quad \delta' x^j \tag{4.34}$$

$$= (\; \delta s_1)^2 + (\; \delta' s)^2$$

where $(\delta s_1)^2 = \bar{g}_{11} (\delta \bar{x}^1)^2 = \pm (\delta \bar{x}^1)^2$ is given by equation (4.29). The new fundamental quadratic form

$$(\; \delta' s)^2 = g_{ij} \quad \delta' x^i \quad \delta' x^j \tag{4.35}$$

is treated in the same way as was the original quadratic form $\delta s^2 = g_{ij} \quad \delta x^i \quad \delta x^j$; a

$$\bar{g}_{22} = \pm 1 \tag{4.36}$$

is obtained. This process is repeated to yield the general result that

$$g_{ii} = \pm 1. \tag{4.37}$$

The table can now be written in the desired diagonal form

$$\bar{g}_{ij} = \quad i \downarrow \begin{array}{c|cccc|c} & 1 & 2 & \cdots & n & \\ \hline & g_{11} & 0 & \cdots & 0 & 1 \\ & \cdot & \bar{g}_{22} & \cdots & 0 & 2 \\ & \cdots & \cdots & \cdots & \cdots & \\ & g_{1n} & 0 & & \bar{g}_{nn} & n \\ \hline \end{array}$$

$$j \longrightarrow$$

$$\bar{g}_{ij} = \begin{array}{c|cccc|c} & 1 & 2 & \cdots & n & \\ \hline i\downarrow & \pm 1 & 0 & \cdots & 0 & 1 \\ & 0 & \pm 1 & \cdots & 0 & 2 \\ & \cdots & \cdots & \cdots & \cdots & \\ & 0 & 0 & \cdots & \pm 1 & n \\ \hline & j \longrightarrow & & & & \end{array} \tag{4.38}$$

It should be remembered that this diagonal form is maintained only in the immediate neighborhood of the point chosen.

5. The divergence and the Laplacian

The divergence is an operator which is applied to a tensor density and not to a tensor. The density divergence of a contravariant vector is formed from the covariant derivative of a contravariant vector density by contraction; the result obtained is a scalar density. If A is a contravariant vector density, its covariant derivative is given by

$$A^k_{;j} = \frac{\partial A^k}{\partial x^j} + \Gamma^k_{mj} A^m - \Gamma^m_{mj} A^k. \tag{2.39}$$

A contraction of indices and change of dummy indices produces

$$A^k_{,k} = \frac{\partial A^k}{x^k} + \Gamma^k_{mk} A^m - \Gamma^k_{km} A^m$$

$$= \frac{\partial A^k}{\partial x^k}. \tag{4.40}$$

The scalar density $\frac{\partial A^k}{\partial x^k}$ is called the divergence of the vector density A^k; it is seen to be independent of the affine connection or the metric tensor.

It was noted in section (IV-3) that $\sqrt{g}$ is a scalar density. If a contravariant vector is multiplied by a scalar density, a contravariant vector density results. The divergence of the vector density may be formed to give the scalar <u>density divergence</u>

$$\frac{\partial}{\partial x^k}\left(\sqrt{g} \quad v^k\right). \tag{4.41}$$

Since $1/\sqrt{g}$ is a scalar capacity, the product of equation (4.41) and $1/\sqrt{g}$ gives an invariant known as the <u>absolute divergence</u>

$$\text{Div } v = 1/\sqrt{g} \quad \frac{\partial}{\partial x^k}\left(\sqrt{g} \quad v^k\right). \tag{4.42}$$

In Cartesian coordinates the absolute divergence is indistinguishable from the density divergence; both reduce to

$$\frac{\partial v^k}{\partial x^k}. \tag{4.43}$$

An interesting application of the divergence to the associated contravariant vector density of the covariant gradient of a scalar function results in the Laplacian. If φ is a scalar function, its gradient is

$$\frac{\partial \varphi}{\partial x^m}. \tag{4.44}$$

From this covariant vector there can be formed the contravariant vector density

$$\sqrt{g} \quad g^{km} \frac{\partial \varphi}{\partial x^m}. \tag{4.45}$$

The absolute divergence of this vector density is known as the Laplacian of φ; it is given by

$$\nabla^2\varphi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g} \quad g^{km} \frac{\partial\varphi}{\partial x^m} \right) \tag{4.46}$$

where ∇^2 represents the operator

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g} \quad g^{km} \frac{\partial}{\partial x^k} \right), \tag{4.47}$$

known as the Laplacian operator. The Laplacian is undefinable without a metric.

The divergence of any tensor density can also be formed by a contraction of the covariant derivative of the tensor density. This result is of little importance; therefore it is omitted.

6. Displacement of units of length in metric geometry

The fundamental quadratic form $ds^2 = g_{ij}\, dx^i\, dx^j$ establishes a standard of orientable length at each point. It now becomes necessary to discuss the displacement of units of length. It is assumed that the units of lengths (unities) e_k are continuous functions of position. The displacement δx represents the vector between two neighboring points P and P'. The length of this vector at P is given by

$$l^2 = g_{ij}\, \delta x^i\, \delta x^j \tag{4.48}$$

If δx is displaced to P', the variation in its magnitude is assumed proportional to the components of the displacement and to its length; that is, the variation

$$\delta l = l\, f_k\, \delta x^k , \tag{4.49}$$

where f_k is a component of a covariant vector proportionality constant. If the displacement δx is carried around a closed path, the necessary condition that the length-gauge have the same value when returned to its starting point is that f_k be a component of the gradient of a scalar function. This conclusion follows from the considerations of sections (III-11) to (III-14). But, if the gauge system of unit vectors is to be the same at all points in space, it is necessary that

$$f_k = 0. \tag{4.50}$$

This condition characterizes <u>Riemannian geometry</u> and the associated Riemannian space.

7. <u>Covariant differentials in Riemannian space</u>

In the previous section Riemannian space was defined as a space for which the same transportable standard of length existed at all points. The existence of the same transportable standard of length permits specification of the coefficients of affine connection in a specialized form. The true increase in a vector u during a parallel transport is given by equation (3.13) as

$$Du^i = du^i + \Gamma^i_{ml}\, u^m\, dx^l \tag{4.51}$$

The particular set of coordinates along which this Du^i is zero is known as geodesic coordinates. Along geodesic coordinates the variation which it is necessary to give to the numerical values of the components to realize parallel transport is given by

$$du^i = -\Gamma^i_{ml}\, u^m\, dx^l \tag{4.52}$$

Since the same standard of length exists at all points, the magnitude of a vector must remain unchanged under transport. This requirement makes it necessary that

$$D\,|u|^2 = D(g_{ij}\, u^i\, u^j) = 0. \tag{4.53}$$

As Du^i and Du^j are zero, it follows that

$$Dg_{ij} = 0. \tag{4.54}$$

This result is a direct consequence of the gauge invariance; it may be used as a criterion for gauge invariance. Further, since

$$Dg_{ij} = dg_{ij} - \Gamma^m_{ik}\, g_{mj}\, dx^k - \Gamma^m_{jk}\, g_{im}\, dx^k = 0, \tag{4.55}$$

therefore

$$Dg_{ij} = \Gamma^m_{ik}\, g_{mn}\, dx^k + \Gamma^m_{jl}\, g_{im}\, dx^k$$

$$= (\Gamma_{j,ik} + \Gamma_{j,jk})\, dx^k, \tag{4.56}$$

where

$$\Gamma_{j,ik} = \Gamma^m_{ik}\, g_{mj}\, . \tag{4.57}$$

and

$$\Gamma_{i,jk} = \Gamma^m_{jk}\, g_{im}\, . \tag{4.58}$$

From these relations the following result is obtained:

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{j,ik} + \Gamma_{i,jk}\, . \tag{4.59}$$

8. <u>The Christoffel symbols</u>

Some interesting consequences result from the relations of the previous section. If the indices of equation (4.59) are permuted and the resulting relations combined by addition and subtraction, the expression

$$\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} = 2\,\Gamma_{k,ij} \tag{4.60}$$

is obtained. The quantity $\Gamma_{k,ij}$ of equation (4.60) is known as Christoffel's three-index symbol of the first kind; it is frequently represented by

$$\Gamma_{k,ij} = [ij,k] \tag{4.61}$$

If the k index of equation (4.61) is raised, the Christoffel symbol of the second kind is obtained; that is, by multiplying $\Gamma_{k,ij}$ by g^{hk} the quantity obtained is

$$g^{hk}\,\Gamma_{h,ij} = \Gamma^k_{ij} = \left\{ {ij \atop k} \right\}. \tag{4.62}$$

where $\left\{ {ij \atop k} \right\}$ is the Christoffel symbol of the second kind. The symbols Γ are more convenient; therefore they will be used on the following pages. The symbols Γ represent affine connections in non-metrical geometry but Christoffel symbols in metrical geometry.

9. Geodesics in Riemannian space

A geodesic line satisfies the following equation [Eq. (3.35)]:

$$\frac{d^2x^i}{d\,t^2} + \Gamma^i_{mk}\,\frac{dx^m}{dt}\,\frac{dx^k}{dt} = 0. \tag{4.63}$$

This equation actually defines a small segment of a straight line. The gradual junction of these small segments represents a geodesic line; therefore the tangent to a geodesic line satisfies the condition of parallel displacement. In Riemannian space the tangent to a geodesic at a point P, after a parallel displacement to a point P' on the geodesic, coincides with the tangent at P'; any vector u making an angle θ with the tangent at P makes the same angle after parallel displacement with the tangent at P'.

The geodesics possess some very interesting properties. They are the lines of "stationary" length; that is, the shortest or longest path between two points will be geodesic. In Euclidean space the geodesics are straight lines. To show that equation (4.63) defines the condition for stationary length, the following variation of the integral of the arc length is formed:

$$\delta S = \delta\int_P^{P'} ds = \delta\int_P^{P'} \sqrt{g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}}\,dt = \delta\int_P^{P'} \dot{s}\,dt = 0, \quad (4.64)$$

where

$$\dot{s}^2 = g_{ij}\,\frac{dx^i}{dt}\,\frac{dx^j}{dt}\,. \quad (4.65)$$

The variation of s can be written as

$$\delta\dot{s} = \frac{\partial\dot{s}}{\partial x^i}\,\delta x^i + \frac{\partial\dot{s}}{\partial\dot{x}^i}\,\delta\dot{x}^i. \quad (4.66)$$

When equation (4.66) is substituted in equation (4.64) and the second term is integrated by parts, the following result is obtained:

$$\delta S = -\int_P^{P'}\left\{\frac{d}{dt}\,\frac{\partial\dot{s}}{\partial x^i} - \frac{\partial s}{\partial x^i}\right\}\delta x^i\,dt\,. \quad (4.67)$$

For this integral to vanish it is necessary that

$$\frac{d}{dt}\left(\frac{\partial s}{\partial x^i}\right) - \frac{\partial s}{\partial x^i} = 0. \qquad (4.68)$$

When s from equation (4.65) is substituted in equation (4.68) and the differentiations are performed, the following equation is obtained:

$$g_{ij}\frac{d^2x^j}{dt^2} + \frac{\partial g_{ij}}{\partial x^m}\frac{dx^m}{dt}\frac{dx^j}{dt} - \frac{1}{2}\frac{\partial g_{lm}}{\partial x^i}\frac{dx^l}{dt}\frac{dx^m}{dt} = 0. \qquad (4.69)$$

If the dummy index j in the second term of equation (4.69) is changed to l, the last two terms can be grouped to give

$$g_{ij}\frac{d^2x}{dt^2} + \left(\frac{\partial g_{il}}{\partial x^m} - \frac{1}{2}\frac{\partial g_{lm}}{\partial x^i}\right)\frac{dx^m}{dt}\frac{dx^l}{dt} = 0. \qquad (8.70)$$

But

$$\frac{\partial g_{il}}{\partial x^m}\frac{dx^m}{dt}\frac{dx^l}{dt} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^m} + \frac{\partial g_{im}}{\partial x^l}\right)\frac{dx^m}{dt}\frac{dx^l}{dt}; \qquad (4.71)$$

therefore equation (4.70) can be written as

$$g_{ij}\frac{d^2x^j}{dt^2} + \frac{1}{2}\left(\frac{\partial g_{il}}{\partial x^m} + \frac{\partial g_{im}}{\partial x^l} - \frac{\partial g_{lm}}{\partial x^i}\right)\frac{dx^m}{dt}\frac{dx^l}{dt} \qquad (4.72)$$

or, by using equation (4.60),

$$g_{ij}\frac{d^2x^j}{dt^2} + \Gamma_{i,ml}\frac{dx^m}{dt}\frac{dx^l}{dt} = 0. \qquad (4.73)$$

Another form of equation (4.73) can be obtained by multiplying equation (4.73) by g^{ik} and then by using equation (4.62); this procedure is represented by

$$g^{ik}\, g_{ij} \frac{d^2x^j}{dt^2} + g^{ik}\, \Gamma_{i,ml} \frac{dx^m}{dt}\frac{dx^l}{dt} = 0.$$

or

$$\delta^k_j \frac{d^2x^j}{dt^2} + \Gamma^k_{ml} \frac{dx^m}{dt}\frac{dx^l}{dt} = 0.$$

The last form reduces to

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ml} \frac{dx^m}{dt}\frac{dx^l}{dt} = 0. \tag{4.74}$$

This equation corresponds to equation (4.63). Since equation (4.63) represents a geodesic, and since equation (4.74) represents a line of stationary length, the line of stationary length must be a geodesic line.

10. Transformation of the geodesic to new coordinates

It is interesting and useful to note how equation (4.74) transforms to some new coordinate system $\bar{x}^i$. The transformation of Γ^k_{ml} is represented by

$$\Gamma^k_{ml} = \frac{1}{2} g^{ik}\left(\frac{\partial g_{il}}{\partial x^m} + \frac{\partial g_{im}}{\partial x^l} - \frac{\partial g_{lm}}{\partial x^i}\right) = \frac{1}{2}\bar{g}^{ab}\frac{\partial x^i}{\partial \bar{x}^a}\frac{\partial x^k}{\partial \bar{x}^b}$$

$$\left[\frac{\partial}{\partial \bar{x}^d}\left(\bar{g}_{ac}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^c}{\partial x^l}\right)\frac{\partial \bar{x}^d}{\partial x^m} + \frac{\partial}{\partial \bar{x}^c}\left(\bar{g}_{ac}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^c}{\partial x^m}\right)\right.$$

$$\left.\frac{\partial \bar{x}^c}{\partial x^l} - \frac{\partial}{\partial \bar{x}^a}\left(\bar{g}_{cd}\frac{\partial \bar{x}^c}{\partial x^l}\frac{\partial \bar{x}^d}{\partial x^m}\right)\frac{\partial \bar{x}^a}{\partial x^i}\right]. \tag{4.75}$$

By considering one of the terms in the brackets in equation (4.75, an expansion formula which is applicable to the remaining terms can be obtained. If the first term is chosen, then its expansion is represented by

$$\frac{\partial}{\partial \bar{x}^d}\left(\bar{g}_{ac}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^c}{\partial x^l}\right)\frac{\partial \bar{x}^d}{\partial x^m} = \frac{\partial \bar{g}_{ac}}{\partial \bar{x}^d}\frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^c}{\partial x^l}\frac{\partial \bar{x}^d}{\partial x^m} +$$

$$\bar{g}_{ac}\left(\frac{\partial^2 \bar{x}^a}{\partial x^m \partial x^i}\frac{\partial \bar{x}^c}{\partial x^l} + \frac{\partial^2 \bar{x}^c}{\partial x^m \partial x^l}\frac{\partial \bar{x}^a}{\partial x^i}\right) \qquad (4.76)$$

The transformation formula now can be reduced to

$$\Gamma^k_{ml} = \bar{\Gamma}^a_{cd}\frac{\partial \bar{x}^c}{\partial x^m}\frac{\partial \bar{x}^d}{\partial x^l}\frac{\partial x^k}{\partial \bar{x}^a} + \frac{\partial^2 \bar{x}^a}{\partial x^m \partial x^l}\frac{\partial x^k}{\partial \bar{x}^a}, \qquad (4.77)$$

where

$$\bar{\Gamma}^a_{cd} = \frac{1}{2}\bar{g}^{ab}\left(\frac{\partial \bar{g}_{bc}}{\partial \bar{x}^d} + \frac{\partial \bar{g}_{bd}}{\partial \bar{x}^c} - \frac{\partial \bar{g}_{cd}}{\partial \bar{x}^b}\right). \qquad (4.78)$$

The presence of the second term in the transformation formula of Γ indicates its non-tensor character.

In order to transform equation (4.74), each term is expressed in terms of the new components and their transformations. The result can be summarized in the equation

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ml}\frac{dx^m}{dt}\frac{dx^l}{dt} = \left(\frac{d^2\bar{x}^a}{dt^2} + \bar{\Gamma}^a_{cd}\frac{d\bar{x}^c}{dt}\frac{d\bar{x}^d}{dt}\right)\frac{\partial x^k}{\partial \bar{x}^a} = 0 \quad (4.79)$$

Equation (4.79) demonstrates that the equation of a geodesic is invariant in form. It can also be interpreted as a demonstration of the fact that the intrinsic derivative operator transforms like a covariant vector.

11. The Riemann-Christoffel curvature tensor

In section (III-14) the non-metrical curvature tensor was defined. For integrability of the displacement of a vector for infinitesimal displacements this tensor must be zero. In metric space the curvature tensors of equations (3.60) and (3.67) assume special forms. The postulate of gauge invariance which characterizes Riemannian space permits expression of these tensors as functions of the derivatives of g_{ik}. The curvature tensors are then completely determined by the g_{ik} and their first and second derivatives. Equation (3.60) can represent the curvature tensor for displacement of a contravariant vectur whose components are u^m or for the associate covariant vector whose components are u_i. For convenience equation (3.60) is repeated as follows:

$$R^i_{m,kh} = \frac{\partial}{\partial x^k} \Gamma^i_{mh} - \frac{\partial}{\partial x^h} \Gamma^i_{mk} + \Gamma^l_{mh} \Gamma^i_{lk} - \Gamma^i_{lh} \Gamma^l_{mk} \qquad (4.80)$$

If the coefficients Γ are considered defined by equations (4.60) and (4.62), equation (4.80) defines the Riemann-Christoffel curvature tensor. The i index of equation (4.80) may be lowered, giving

$$R_{jm,kh} = g_{ij} R^i_{m,kh} = g_{ij}\left(\frac{\partial}{\partial x^k} \Gamma^i_{mh} - \frac{\partial}{\partial x^h} \Gamma^i_{mk} + \Gamma^l_{mh} \Gamma^i_{lk} - \Gamma^i_{lk} \Gamma^l_{mk}\right) \qquad (4.81)$$

Now some special relationships can be shown to exist between the components of the fourth order Riemann-Christoffel tensor. The first term in equation (4.81) can be written in another form.

$$g^{ij}\frac{\partial}{\partial x^k}\Gamma^i_{mh} = \frac{\partial}{\partial x^k}(g_{ij}\Gamma^i_{mh}) - \Gamma^i_{mh}\frac{\partial g_{ij}}{\partial x^k}$$

$$= \frac{\partial}{\partial x^k}\Gamma_{j,mh} - \Gamma^i_{mh}\left(\Gamma^i_{jk} + \Gamma^j_{ik}\right), \quad (4.82)$$

where, by equation (4.59),

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^i_{jk} + \Gamma^j_{ik} \quad (4.83)$$

Similarly, the second term in equation (4.81) can be written in the form

$$g^{ij}\frac{\partial}{\partial x^h}\Gamma^i_{mk} = \frac{\partial}{\partial x^h}\Gamma_{j,mk} - \Gamma^i_{mk}\left(\Gamma^i_{jh} + \Gamma^j_{ih}\right). \quad (4.84)$$

By substituting the above results into equation (4.81), the following form of Riemann-Christoffel curvature tensor is obtained:

$$R_{jm,kh} = \frac{\partial}{\partial x^k}\Gamma_{j,mh} - \Gamma^i_{mh}\left(\Gamma^i_{jk} + \Gamma^j_{ik}\right)$$
$$+ \frac{\partial}{\partial x^h}\Gamma_{j,mk} - \Gamma^i_{mk}\left(\Gamma^i_{jh} + \Gamma^j_{ih}\right)$$
$$+ \Gamma_{j,lk}\Gamma^l_{mh} - \Gamma_{j,mh}\Gamma^l_{mh} \quad (4.85)$$

By using equations (4.60) and (4.62, equation (4.85 can be written as

$$R_{jm,kh} = \frac{1}{2}\left(\frac{\partial^2 g_{jh}}{\partial x^m \partial x^k} + \frac{\partial^2 g_{mk}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{jk}}{\partial x^m \partial x^h} - \frac{\partial^2 g_{mh}}{\partial x^j \partial x^k}\right)$$
$$+ g^{nP}\left(\Gamma_{n,mk}\Gamma_{P,jh} - \Gamma_{n,mh}\Gamma_{P,jk}\right) \quad (4.86)$$

From this equation it follows that $R_{jm,kh}$ possesses the following symmetries:

$$R_{jm,kh} = R_{mj,kh},$$
$$R_{jm,kh} = - R_{jm,hk},$$
$$R_{jm,kh} = R_{kh,jm}. \qquad (4.87)$$

The first two of these equations express the fact that $R_{jm,kh}$ is anti-symmetric in j,m and s,t; the last equation expresses the symmetry between the two groups of indices jm and kh.

Since the coefficients Γ are constants in Euclidean space with Cartesian reference frames, it follows that the Riemannian-Christoffel tensor is zero for this case.

12. The contracted tensor of Ricci and Einstein

The mixed Riemann-Christoffel fourth order tensor $g^{ij} R_{jm,kh}$ can be contracted with respect to the i and h indices. The resulting second order tensor

$$R_{m,k} = g^{hj} R_{jm,kh} = \frac{\partial}{\partial x^k} \Gamma^h_{mh} - \frac{\partial}{\partial x^h} \Gamma^h_{mk} + \Gamma^h_{lk} \Gamma^l_{mh} - \Gamma^h_{lh} \Gamma^l_{mk} \qquad (4.88)$$

is called the contracted curvature tensor of Ricci and Einstein; when the components of g are constants, the first and last terms vanish. This tensor was used by Einstein to represent the gravitational field equations outside of matter.

The tensor R_{mk} can easily be written as a mixed tensor by multiplying it by g^{im}. If the mixed tensor is contracted to give a true scalar, then the constant of curvature

$$R = g^{km} R_{mk} \tag{4.89}$$

is obtained. This constant of curvature has an important role in the theory of relativity.

13. The identities of Bianchi

There are some identities between the covariant partial derivatives of the curvature tensor, known as the Bianchi identities. These identities are given by

$$\frac{D}{Dx^h} R_{ij,kl} + \frac{D}{Dx^k} R_{ij,lh} + \frac{D}{Dx^l} R_{ij,hk} = 0 \quad . \tag{4.90}$$

By using a system of geodesic coordinates the tensor R, expressed in the form given by equation (4.86) at the point considered, these identities are easily verified. This procedure eliminates the partial derivatives of the Christoffel symbols by annulling the first derivatives of the g; it reduces locally the covariant derivatives to ordinary derivatives. Once the identity is verified for one system of axes it is true for any other system.

14. Normal coordinates of Riemann

A coordinate system is called an affine normal coordinate system in case the solutions of the differential equations for geodesics, which represent paths through the origin, take the linear form.

In a Riemannian space, a local Euclidean space can be associated, to a first degree of approximation, with a given point in the immediate neighborhood of that point. An orthoginal system of reference may just as well be used. In this case the fundamental quadratic form reduces to the form

$$ds^2 = (dv^1)^2 + (dv^2)^2 + ... (dv^n)^2, \tag{4.91}$$

where the v^k represents the new coordinates. If s is a length measured along a geodesic issuing from the given point and l^k the direction cosine of the geodesic, the coordinates of a nearby point, to a first order of approximation, are given by

$$v^k = l^k s \tag{4.92}$$

The equation of a geodesic for such coordinates is given by

$$\frac{d^2v^k}{ds^2} + \Gamma^k_{lm} \frac{dv^l}{ds} \frac{dv^m}{ds} = 0. \tag{4.93}$$

Since v^k is proportional to s, the last form of equation can just as well be written in the form

$$\Gamma^k_{lm} \; v^l \, v^m = 0. \tag{4.94}$$

As these identities have their loci all along the geodesics, the partial derivatives with respect to s of equation (4.94) are zero; that is,

$$\frac{\partial \Gamma k,lm}{\partial v^n} \; v^l \, v^m \, v^n = 0. \tag{4.95}$$

Hence, for arbitrary values of v, the coefficient of $v^l \, v^m \, v^n$ is zero; therefore

$$\frac{\partial \Gamma k,lm}{\partial v^n} + \frac{\partial \Gamma k,mn}{\partial v^l} + \frac{\partial \Gamma k,nl}{\partial v^m} = 0. \tag{4.96}$$

Near the given point the Γ are infinitely small, but not their partial derivatives; therefore the curvature tensor of equation (4.85) can assume the following simple form in the new coordinates:

$$R_{kl,mn} = \frac{\partial \Gamma_{k,ln}}{\partial v^m} - \frac{\partial \Gamma_{k,lm}}{\partial v^n} . \tag{4.97}$$

The last two systems of equations can be solved for the $\dfrac{\partial \Gamma_{k,lm}}{\partial v^n}$ in terms of the R and the coordinates, yielding

$$\frac{\partial \Gamma_{k,lm}}{\partial v^n} = \frac{1}{3} (R_{kl,mn} + R_{km,nl}) \tag{4.98}$$

For the assumptions made, this equation can be replaced by

$$\Gamma_{k,lm} = \frac{1}{3} (R_{kl,mn} + R_{km,nl})\, v^n \tag{4.99}$$

in the immediate neighborhood of the point.

By making use of equations (4.59) and (4.99), there can be obtained for the special coordinates v^i

$$\frac{\partial g_{kl}}{\partial v^m} = \Gamma_{k,lm} + \Gamma_{l,km} = \frac{1}{3} (R_{kl,mn} + R_{km,nl})\, v^n \tag{4.100}$$

which becomes, upon integration,

$$g_{kl} = \delta_{kl} + \frac{1}{3} (R_{km,nl} + R_{lm,nk})\, v^m v^n, \tag{4.101}$$

to a first degree of approximation.

The following important conclusion is obtained from the above results: if the curvature tensor is not zero the space cannot be Euclidean, but to a first degree of approximation it is Euclidean. The generalized theory of relativity utilizes successive approximations of the kind just described.

V. INTRINSIC TENSOR ANALYSIS

In the usual tensor analysis of the coordinates, tensor components are referred to the differentials of the coordinates, dx^k, pertaining to the ennuple of parametric curves; they may be referred, however, to the differentials of arc, ds^a, of the curves of an arbitrary ennuple E (8). In the latter case, the tensor analysis is called intrinsic tensor analysis. In this tensor analysis the components of tensors are invariants.

In the intrinsic theory there is no need of actual coordinates pertaining to the ennuple E; the differentials of arc suffice. The ennuple E, therefore, does not have to be an ennuple with which coordinates can be associated. It will be assumed in what follows that the ennuple does not have to be an ennuple with which coordinates can be associated.

From a second point of view, the intrinsic tensor analysis may be considered as the result of employing the arcs of the curves of the arbitrary ennuple E as non-holonomic parameters. Vranceanu (23) used non-holonomic parameters in the study of general connections and non-holonomic manifolds.

1. Systems of congruences

If β^i are the n continuous and differentiable components of a contravariant unit vector β, a displacement in the direction of the vector at a point $P(x^i)$ satisfies the equations

$$\frac{dx^1}{\beta^1} = \frac{dx^2}{\beta^2} = \cdots\cdots\cdots\cdots \frac{dx^3}{\beta^n} \tag{5.1}$$

These differential equations admit n - 1 independent solutions

$$\phi^j(x^1, \ldots x^n) = c^n \quad (j = 1, \ldots, n - 1), \tag{5.2}$$

where the c's are arbitrary constants and the matrix $\left[\frac{\partial \phi^j}{\partial x^i}\right]$ is of rank n - 1. Equations (5.1) are said to define a congruence of curves. Through each point P where the quantities β^i are not all zero, there passes one curve of the congruence and only one, the tangent to the curve at P; this tangent has the same direction as the vector β at this point.

If n contravariant unit vectors β_α $(\alpha = 1, \ldots, n)$ are to be independent, they must satisfy the determinant

$$\left|\beta_\alpha^{\ i}\right| \neq 0 \tag{5.3}$$

at least in a certain region of the space. These n vectors determine a system of n independent congruences, known as an ennuple, in such a way that through each point P there passes n curves of the system having as tangents at P the directions of the n independent unit vectors passing through this point.

Since the determinant $\left|\beta_\alpha^{\ i}\right|$ is different from zero, it can be expanded by minors. That is,

$$\left|\beta_\alpha^i\right| = \sum_i \beta_\alpha^i \times \text{cofactor of } \beta_\alpha^i \text{ in } \left|\beta_\alpha^i\right| \tag{5.4}$$

If both sides of equation (5.4) are divided by $\left|\beta_\alpha^{\ i}\right|$, the result is:

$$1 = \sum \beta_\alpha^{\ i}\, \beta^\alpha_{\ i} \,; \quad 1 = \sum_i \beta_\alpha^{\ i}\, \beta^\alpha_{\ i} \tag{5.5}$$

where $\beta^{\alpha}_{\ i}$ is the cofactor of $\beta_{\alpha}^{\ i}$ in the determinant $|\beta_{\alpha}^{\ i}|$ divided by $|\beta_{\alpha}^{\ i}|$. Since any β^{α}_{j} is orthogonal to β_{α}^{i} for $x \neq j$, it follows that

$$\beta_{\alpha}^{i} \quad \beta^{\alpha}_{j} = \delta^{i}_{j} ; \quad \beta^{a}_{\alpha} \ \beta^{\alpha}_{b} = \delta^{a}_{b} . \tag{5.6}$$

These relations demonstrate that β^{α}_{1}, β^{α}_{2}, ... β^{α}_{n} can be considered as the components of n covariant unit vectors β^{α}. These n vectors are known as the vectors conjugate (or reciprocal) to the n unit vectors tangent to the curves of the ennuple E.

The special case of n mutually orthogonal non-null vector fields in an n-space is called an orthogonal ennuple.

2. Transformations to intrinsic components

The transformation from the ordinary to the intrinsic components of a tensor obeys the standard formal laws of tensor analysis. If a small displacement vector dx at $P(x^i)$ is transformed or projected on the congruences, its components along the congruences are given by

$$ds^a = \beta^{a}_{i} \ dx^i ; \tag{5.7}$$

its components ds^a along the congruences projected on the coordinate curves are given by

$$dx^i = \beta^{i}_{a} \ ds^a . \tag{5.8}$$

From the last two sets of equations, it follows that

$$\beta^{a}_{i} = \frac{\partial s^a}{\partial x^i}, \text{ and } \quad \beta^{i}_{a} = \frac{\partial x^i}{\partial s^a} . \tag{5.9}$$

If, in the transformation of the components of a tensor which is the result of a change from coordinates x^i to $\bar{x}^i$, $\bar{x}^i$ is replaced by s^i, the transformation becomes that from the ordinary to the intrinsic components. The law of transformation from the ordinary to the intrinsic components of a general tensor is, using (2.16) and (5.9),

$$T^{a_1 \cdots a_n}_{b_1 \cdots b_m} = \frac{\partial s^{a_1}}{\partial x^{i_1}} \cdots \frac{\partial s^{a_n}}{\partial x^{i_n}} \frac{\partial x^{j_1}}{\partial s^{b_1}} \cdots \frac{\partial x^{j_m}}{\partial s^{b_m}} T^{i_1 \cdots i_n}_{j_1 \cdots j_m}, \quad (5.10)$$

where $T^{a_1 \cdots a_n}_{b_1 \cdots b_m}$ is a component of T along the congruences.

$\frac{\partial}{\partial s^i}$ denotes directional differentiation in the positive direction of an arbitrary curve of the congruences. The relations between the directional derivatives $\frac{\partial}{\partial s^i}$ and the partial derivatives $\frac{\partial}{\partial x^i}$ are obviously

$$\frac{\partial f}{\partial s^i} = \frac{\partial x^j}{\partial s^i} \frac{\partial f}{\partial x^j}, \quad \frac{\partial f}{\partial x^i} = \frac{\partial s^j}{\partial x^i} \frac{\partial f}{\partial s^j}. \quad (5.11)$$

From formulas (5.11) it follows that

$$df = \frac{\partial f}{\partial s^i} ds^i. \quad (5.12)$$

In the transformations of non-tensor objects such as the coefficients of connections and the Christoffel symbols the order of partial differentiation must be closely observed. The non-permutability of partial differentiation often introduces additional terms in the transformation of non-tensor objects.

3. Integrability conditions

The integrability conditions for the parallel displacement of a scalar quantity require that [Sect. (III-13)]

$$\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} = \frac{\partial^2 \varphi}{\partial x^j \partial x^i} - \frac{\partial^2 \varphi}{\partial x^i \partial x^j} = 0; \tag{5.13}$$

that is, the order of partial differentiation of the scalar quantity should not affect the result obtained. Equation (5.13) can be expressed in terms of the directional derivatives by means of equations (5.11). In order to obtain the resulting integrability conditions in the indices i and j, equations (5.13) are first rewritten as

$$\frac{\partial^2 \varphi}{\partial x^q \partial x^p} - \frac{\partial^2 \varphi}{\partial x^p \partial x^q} = 0. \tag{5.14}$$

Now when equations (5.11) are substituted in equation (5.14), the expression

$$\frac{\partial^2 \varphi}{\partial x^q \partial x^p} - \frac{\partial^2 \varphi}{\partial x^p \partial x^q} = \frac{\partial}{\partial s^j} \left(\frac{\partial \varphi}{\partial s^i} \frac{\partial s^i}{\partial x^p} \right) \frac{\partial s^j}{\partial x^q} - \frac{\partial}{\partial s^i} \left(\frac{\partial \varphi}{\partial s^j} \frac{\partial s^j}{\partial x^q} \right) \frac{\partial s^i}{\partial x^p}, \tag{5.15}$$

is obtained. If equations (5.15) are expanded and multiplied by

$$\frac{\partial x^r}{\partial s^i} \frac{\partial x^r}{\partial s^j},$$

the result is that

$$\left(\frac{\partial}{\partial s^j} \frac{\partial \varphi}{\partial s^i} - \frac{\partial}{\partial s^i} \frac{\partial \varphi}{\partial s^j} \right) \delta^r_p \quad \delta^r_q + \left[\left(\frac{\partial}{\partial s^j} \frac{\partial s^k}{\partial x^p} \right) \delta^r_q \frac{x^r}{s^i} - \left(\frac{\partial}{\partial s^i} \frac{\partial s^k}{\partial x^q} \right) \delta^r_p \frac{\partial x^r}{\partial s^j} \right] \frac{\varphi}{x^k} = 0. \tag{5.16}$$

All the terms must vanish except for $p = q = r$; therefore the conditions of equation (5.16) are expressed equally as well by

$$\frac{\partial}{\partial s^j}\frac{\partial \varphi}{\partial s^i} - \frac{\partial}{\partial s^i}\frac{\partial \varphi}{\partial s^j} = B^k_{ij}\frac{\partial \varphi}{\partial s^k}, \qquad (5.17)$$

where

$$B^k_{ij} = \frac{\partial x^r}{\partial s^j}\left(\frac{\partial}{\partial s^i}\frac{\partial s^k}{\partial x^r}\right) - \frac{\partial x^r}{\partial s^i}\left(\frac{\partial}{\partial s^j}\frac{\partial s^k}{\partial x^r}\right)$$

$$= \frac{\partial s^k}{\partial x^r}\left(\frac{\partial}{\partial s^j}\frac{\partial x^r}{\partial s^i} - \frac{\partial}{\partial s^i}\frac{\partial x^r}{\partial s^j}\right). \qquad (5.18)$$

These relations obviously give the necessary and sufficient conditions that $\varphi_k\, ds^k$, where $\varphi_k = \varphi_k(x^i)$, be an exact differential (8, p. 54.6).

4. Metric connections

Up to this time no consideration has been given to coefficients of connections other than the symmetric type. With the introduction of a metric and gauge invariance the basis of Riemannian geometry was determined. With the introduction of non-holonomic transformation coefficients, that is coefficients for which $\frac{\partial}{\partial x^j}\frac{\partial s^a}{\partial x^i} \neq \frac{\partial}{\partial x^i}\frac{\partial s^a}{\partial x^j}$, the coefficients of connection are not symmetric; therefore $L^i_{mk} \neq L^i_{km}$. When a metric is introduced, the condition that the covariant derivative of the metric vanish, still must be satisfied; but since the symmetry conditions of the coefficients of connection do not exist, the reduction of them to Christoffel symbols is not possible. The covariant derivative of the metric tensor with respect to an ennuple is given by

$$\frac{Dg_{ij}}{Ds^k} = \frac{\partial g_{ij}}{\partial s^k} - g_{i\alpha} L^{\alpha}_{jk} - g_{\alpha j} L^{\alpha}_{ik} \tag{5.19}$$

$$= \frac{\partial g_{ij}}{\partial s^k} - L_{i,jk} - L_{j,ik},$$

where $L_{i,jk}$ and $L_{j,ik}$ are coefficients of connection. From the considerations of equations (4.53) it follows that

$$D\,|u|^2 = \left(\frac{\partial g_{ij}}{\partial s^k} - L_{i,jk} - L_{j,ik}\right) u^i\, u^j ds^k = 0. \tag{5.20}$$

By the quotient law the quantity $\left(\frac{\partial g_{ij}}{\partial s^k} - L_{i,jk} - L_{j,ik}\right)$ must be a tensor of the third order; this tensor is denoted by $2\,B_{k,ij}$. Then

$$2\,B_{k,ij} = \frac{\partial g_{ij}}{\partial s^k} - L_{i,jk} - L_{j,ik}\,. \tag{5.21}$$

Similarly

$$2\,B_{j,ki} = \frac{\partial g_{ik}}{\partial s^j} - L_{i,kj} - L_{k,ij},$$

and

$$2\,B_{i,jk} = \frac{\partial g_{jk}}{\partial s^i} - L_{j,ik} - L_{k,ji}\,.$$

By adding the last two equations and subtracting equation (5.21) the following tensor is obtained:

$$B_{j,ki} + B_{i,jk} - B_{k,ij} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial s^j} + \frac{\partial g_{jk}}{\partial s^i} - \frac{\partial g_{ij}}{\partial s^k}\right) - L_{k,ij}. \tag{5.22}$$

Equation (5.22) can also be written as

$$S_{k,ij} = L_{k,ij} - \Gamma_{k,ij}\,, \tag{5.23}$$

where

$$S_{k,ij} = B_{k,ij} - B_{j,ki} - B_{i,jk} \tag{5.24}$$

is a tensor, and

$$\Gamma_{k,ij} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial s^j} + \frac{\partial g_{jk}}{\partial s^i} - \frac{\partial g_{ij}}{\partial s^k}\right), \tag{5.25}$$

is the Christoffel symbol of the first kind. $S_{k,ij}$ is actually a component of the <u>torsion tensor</u>; it can be shown to be equal to the anti-symmetric part of the components of the coefficients of connection. By means of equation (5.23) and the symmetry of the Christoffel symbols,

$$S_{k,ij} = \frac{1}{2}(L_{k,ij} - L_{k,ji}) = \frac{1}{2}(\Gamma_{k,ij} + S_{k,ij} - \Gamma_{k,ji} - S_{k,ji}),$$

or

$$S_{k,ij} = \frac{1}{2}(S_{k,ij} - S_{k,ji}) \tag{5.26}$$

an anti-symmetric tensor in the indices ij.

It should be noted that the torsion tensor and the coefficients of connection have the same transformation laws for congruence transformations as for coordinate transformations. Another interesting point is that the space corresponding to the coefficients of connection can be Riemannian only if the torsion tensor is null.

5. <u>Geodesics in terms of the congruences</u>

If a directed curve C is given by $x^i = x^i(s)$, where s is the parametric arc length, and if $\bar{a}^i$ and a^i are the ordinary and intrinsic components, respectively, of the unit vector tangent to C at an arbitrary point P of C, then

$$dx^i = \bar{a}^i \, ds. \tag{5.27}$$

But

$$dx^i = \frac{\partial x^i}{\partial s^j} ds^j; \tag{5.28}$$

therefore

$$\frac{\partial x^i}{\partial s^j} ds^j = \bar{a}^i \, ds. \tag{5.29}$$

Since

$$\bar{a}^i = a^j \frac{\partial x^i}{\partial s^j}, \tag{5.30}$$

equation (5.29) becomes

$$\frac{\partial x^i}{\partial s^j} ds^j = a^j \frac{\partial x^i}{\partial s^j} ds,$$

or

$$ds^j = a^j \, ds, \tag{5.31}$$

from which

$$a^j = \frac{ds^j}{ds} \tag{5.32}$$

The equation of a geodesic in Riemannian space with coordinate variables was given in a form equivalent to

$$\frac{D}{Ds} \bar{a}^i = \frac{d\bar{a}^i}{ds} + \Gamma^i_{\alpha\beta} \; \bar{a}^\alpha \; \bar{a}^\beta, \tag{5.33}$$

where $\bar{a}^i$ is given by equation (5.27). But since the first intrinsic derivative of a tensor quantity is another tensor of one higher covariance order, equation (5.33) transforms to another coordinate system as follows:

$$\frac{D}{Ds} \bar{a}^i = \frac{\partial x^i}{\partial x'^j} \; \frac{D}{Ds} a'^j. \tag{5.34}$$

But a transformation to congruences obeys the same law; therefore

$$\frac{D}{Ds} \bar{a}^i = \frac{\partial x^i}{\partial s^j} \frac{D}{Ds} a^j, \tag{5.35}$$

where a^j is given by equation (5.32). In terms of the congruences the geodesic equation is defined by the transformation to an ennuple of congruences of the geodesic equation in terms of the coordinates; that is, the geodesic equation in terms of the congruences is given by

$$\frac{D}{Ds}\frac{dx^i}{ds} = \frac{D}{Ds}\left(\frac{ds^j}{ds}\right)\frac{\partial x^i}{\partial s^j} = 0$$
$$= \left(\frac{d^2s^j}{ds^2} + L^j_{ml}\frac{ds^m}{ds}\frac{ds^l}{ds}\right)\frac{\partial x^i}{\partial s^j} = 0, \tag{5.36}$$

where, by equation (5.23),

$$L^j_{ml} = \Gamma^j_{ml} - S^j_{ml} \tag{5.37}$$

are the anti-symmetric coefficients of connections resulting from the transformation. It so happens that with respect to the coordinates the coefficients of connection are reduced to the Christoffel symbols; but the transformation of the covariant derivative to an arbitrary ennuple of congruences introduces the torsion tensor S.

6. Transformations from one ennuple of congruences to a second

In the previous section it was shown that if a curve C is given by $x^i = x^i(s)$ that the ordinary and intrinsic components $\overline{a}^i$ and a^i of the unit vector tangent to C are given by

$$\overline{a}^i = \frac{dx^i}{ds}$$
and
$$a^i = \frac{ds^i}{ds}, \tag{5.38}$$

respectively. It follows that

$$\frac{dx^i}{ds} = \frac{ds^j}{ds}\frac{\partial x^i}{\partial s^j}, \quad \frac{ds^i}{ds} = \frac{dx^j}{ds}\frac{\partial s^i}{\partial x^j} \tag{5.39}$$

with respect to some ennuple E. Similar formulas hold when C is referred to another ennuple E' instead of E.

The relation $a^i = \frac{ds^i}{ds}$ suggests, following Graustein (8, p. 572), representing the contravariant components, referred to E, of the unit vector tangent to the general curve $C_{h'}$ of the hth congruence of E' by

$$a_{h'|}{}^{i} = \frac{\partial s^i}{\partial s'^h}, \tag{5.40}$$

and the contravariant components, referred to E', of the unit vector tangent to the general curve C_h of the hth congruence of E by

$$a_{h|}{}^{i} = \frac{\partial s'^i}{\partial s^h}. \tag{5.41}$$

Then by applying the principles of equation (5.39), the following relations are obtained:

$$\begin{aligned} \frac{\partial x^i}{\partial s'^h} &= \frac{\partial s^j}{\partial s'^h}\frac{\partial x^i}{\partial s^j}, & \qquad \frac{\partial s^i}{\partial s'^h} &= \frac{\partial x^j}{\partial x'^h}\frac{\partial s^i}{\partial x^j}, \\ \frac{\partial x^i}{\partial s^h} &= \frac{\partial s'^j}{\partial s^h}\frac{\partial x^i}{\partial s'^j}, & \qquad \frac{\partial s'^i}{\partial s^h} &= \frac{\partial x^j}{\partial s^h}\frac{\partial s'^i}{\partial x^j}. \end{aligned} \tag{5.42}$$

By means of the principles of equations (5.6) and (5.9), it follows from equation (5.42) that

$$\frac{\partial s^j}{\partial s'^h}\frac{\partial x^i}{\partial s^j}\frac{\partial s'^k}{\partial x^i} = \frac{\partial s'^i}{\partial x^i}\frac{\partial x^i}{\partial s'^h}, \tag{5.43}$$

or

$$\frac{\partial s^j}{\partial s'^h}\frac{\partial s'^k}{\partial s^j} = \delta_h^k.$$

In a similar way it is possible to obtain the relation

$$\frac{\partial s^i}{\partial s'^j}\frac{\partial s'^j}{\partial s^k} = \delta^i_k\,. \tag{5.44}$$

Equations (5.43) and (5.44) state that the systems of quantities, $\dfrac{\partial s^j}{\partial s'^i}$ and $\dfrac{\partial s'^i}{\partial s^j}$ are conjugate to one another.

According to equation (5.43),

$$\delta^k_h = \frac{\partial s^j}{\partial s'^h}\frac{\partial s'^k}{\partial s^j}\,, \tag{5.45}$$

therefore

$$\begin{aligned} \delta^k_h\, ds'^h &= \frac{\partial s'^k}{\partial s^j}\, ds^j, \\ ds'^k &= \frac{\partial s'^k}{\partial s^j}\, ds^j. \end{aligned} \tag{5.46}$$

In a similar way the relation

$$ds^i = \frac{\partial s^i}{\partial s'^j}\, ds'^j \tag{5.47}$$

can be obtained.

The partial derivatives of a function with respect to two sets of variables s'^i and s^i are related thus:

$$\frac{\partial f}{\partial s'^i} = \frac{\partial f}{\partial s^\alpha}\frac{\partial s^\alpha}{\partial s'^i}\,; \quad \frac{\partial f}{\partial s^i} = \frac{\partial f}{\partial s'^\alpha}\frac{\partial s'^\alpha}{\partial s^i}\,. \tag{5.48}$$

These equations define the relations between the directional derivatives.

Equations (5.46), (5.47), and (5.48) show that <u>the transformation</u>

from the components of a tensor, referred to an ennuple E_1 to the components, referred to E', obeys the usual laws of tensor analysis; for example, the components of a general tensor T transform by the rule

$$T'^{a_1 \cdots a_n}_{b_1 \cdots b_m} = \frac{\partial s^{j_1}}{\partial s'^{b_1}} \cdots \frac{\partial s^{j_m}}{\partial s'^{b_m}} \frac{\partial s'^{a_1}}{\partial s^{i_1}} \cdots \frac{\partial s'^{a_n}}{\partial s^{i_n}}$$

$$T^{i_1 \cdots i_n}_{j_1 \cdots j_m} . \qquad (5.49)$$

VI. VECTOR ANALYSIS RELATED TO TENSOR ANALYSIS

Vector analysis can be considered as a specialized branch of the metrical tensor analysis for zero, first, and second order tensors in three-dimensional space. Being a specialized branch of tensor analysis, vector analysis differs from tensor analysis in several important aspects. It is based upon an intuitive definition of a vector, the representation of second order anti-symmetric tensors as pseudo-vectors, the use of the symbolism of the "del" operator ($\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$), and is limited to three-dimensional Euclidean space with identical and unchanging unit lengths along the various reference axes. These specializations simplify the mathematical procedures of the vector calculus, but result in less rigor, more confusion, and added complication for the study of the more general tensor analysis.

Roughly speaking, in vector analysis a vector is defined as "a quantity which has direction and magnitude." This definition fails, however, to completely characterize a vector; it is not sufficiently clear and comprehensive. As Murnaghan points out (15, p. 2), it fails to explain what is meant by "having direction." As examples of lack of clarity of such a definition of a vector, the vector product of two vectors is non-commutative; the curl of a vector depends upon the sense of rotation of the reference axes; the representation of rotation by vectors sometimes leads to untrue results. The vectors which obey the usual algebraic laws of tensor analysis are called "polar" vectors; they are true first order tensors. Vectors such as the vector product and the curl are "axial" vectors. The axial vectors are not true vectors but are pseudo-vectors formed from true second order anti-symmetric tensors. The axial vectors are sensitive to the sense of rotation of the reference axes in a transformation of Cartesian axes; they behave as usual vectors as long as the same sense of rotation of the reference axes (either "right-handed" or "left-handed" reference systems) is maintained; but, if the sense of rotation is changed, the axial vectors change sign.

In this chapter an attempt is made to clarify some of the mathematical foundations of vector analysis in their relationships to tensor analysis by using the relationships between pseudo-tensors and tensors.

1. Second order anti-symmetric tensors as axial vectors

The axial vectors of vector analysis are derived from second order anti-symmetric tensors in three dimensional space. In three dimensional vector space the components t_{ij} of a second order anti-symmetric tensor can be arranged in the following array:

$$t_{ij} = \begin{array}{c|ccc|c} & 1 & 2 & 3 & \\ i \downarrow & t_{11} & t_{12} & t_{13} & 1 \\ & t_{21} & t_{22} & t_{23} & 2 \\ & t_{31} & t_{32} & t_{33} & 3 \\ & & j \rightarrow & & \end{array} \qquad (6.1)$$

But, since $t_{ij} = -t_{ji}$, equation (6.1) can be written in the form

$$t_{ij} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ i\downarrow & \left|\begin{array}{ccc} 0 & t_{12} & t_{13} \\ -t_{12} & 0 & t_{23} \\ -t_{13} & -t_{23} & 0 \end{array}\right| \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ & j \longrightarrow \end{array} \qquad (6.2)$$

with only three independent components. If a choice of the order of the indices (the order 1, 2, 3 for example), which amounts to choosing a privileged sense of rotation of the axes (right-handed system) is made, then with each component t_{ij} a component T^k can be associated by choosing the indices so that the order i, j, k is deduced from the order 1, 2, 3 by an even permutation. The table (6.2) can now be written as

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ i\downarrow & \left|\begin{array}{ccc} 0 & T^3 & -T^2 \\ -T^3 & 0 & T^1 \\ T^2 & -T^1 & 0 \end{array}\right| \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ & j \longrightarrow \end{array} \qquad (6.3)$$

a pseudo-tensor.

The cause of the commutative addition and multiplication troubles is easily understood after a transformation of the components of (6.2) and (6.3) to a new reference system is made. In a new coordinate system $\overline{x}^i$ the components of the tensor t are $\overline{t}_{ab}$. Then, since $\overline{t}_{ab}$ can be represented by $\overline{T}^c$ with a, b, c, an even permutation of the order 1, 2, 3, the transformation formulas for the $\overline{t}_{ab}$ become

$$t_{ij} = \frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}\bar{t}_{ab} = \frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial \bar{x}^b}{\partial x^j}\;\bar{T}^c \tag{6.4}$$

or, by expanding and regrouping,

$$\begin{aligned} t_{ij} = &\left(\frac{\partial \bar{x}^2}{\partial x^i}\frac{\partial \bar{x}^3}{\partial x^j} - \frac{\partial \bar{x}^3}{\partial x^i}\frac{\partial \bar{x}^2}{\partial x^j}\right)\bar{T}^1 \\ &+ \left(\frac{\partial \bar{x}^3}{\partial x^i}\frac{\partial \bar{x}^1}{\partial x^j} - \frac{\partial \bar{x}^1}{\partial x^i}\frac{\partial \bar{x}^3}{\partial x^j}\right)\bar{T}^2 \\ &+ \left(\frac{\partial \bar{x}^1}{\partial x^i}\frac{\partial \bar{x}^2}{\partial x^j} - \frac{\partial \bar{x}^2}{\partial x^i}\frac{\partial \bar{x}^1}{\partial x^j}\right)\bar{T}^3 \end{aligned} \tag{6.5}$$

The coefficients of the $\bar{T}$ are clearly the minors of the determinant

$$\Delta^{-1} = \begin{vmatrix} \frac{\partial \bar{x}^1}{\partial x^i} & \frac{\partial \bar{x}^2}{\partial x^i} & \frac{\partial \bar{x}^3}{\partial x^i} \\ \frac{\partial \bar{x}^1}{\partial x^j} & \frac{\partial \bar{x}^2}{\partial x^j} & \frac{\partial \bar{x}^3}{\partial x^j} \\ \frac{\partial \bar{x}^1}{\partial x^k} & \frac{\partial \bar{x}^2}{\partial x^k} & \frac{\partial \bar{x}^3}{\partial x^k} \end{vmatrix} = \sum_{l,m,n} \pm \frac{\partial \bar{x}^l}{\partial x^k}\;\frac{\partial \bar{x}^m}{\partial x^i}\frac{\partial \bar{x}^n}{\partial x^j}\,, \tag{6.6}$$

where the order i,j,k is deduced from the order 1, 2, 3 by an even permutation. The sign of the terms in the summation is made + or - depending upon whether l, m, n is an even or an odd permutation of the 1, 2, 3. Solving equation (6.6) for its minors, it is possible to obtain

$$\sum_{m,n} \pm \frac{\partial \bar{x}^m}{\partial x^i}\;\frac{\partial \bar{x}^n}{\partial x^j} = \sum_{l} \Delta^{-1}\,\frac{\partial x^k}{\partial \bar{x}^l} \tag{6.7}$$

These minors can be substituted into equation (6.5) to obtain

$$t_{ij} = \Delta^{-1}\left(\frac{\partial x^k}{\partial \bar{x}^1}\bar{T}^1 + \frac{\partial x^k}{\partial \bar{x}^2}\bar{T}^2 + \frac{\partial x^k}{\partial \bar{x}^3}\bar{T}^3\right) = T^k, \qquad (6.8)$$

a vector density. This is the transformation formula of the T; it is clearly not that of a tensor. If it were not for the factor Δ^{-1}, however, this transformation formula would be that of a true first order tensor. It so happens that for rectangular coordinate systems the determinant of the direction cosines has a value of +1 or -1 depending upon whether the two sets of axes have the same or opposite sense of rotation; therefore as long as the same sense of rotation is maintained, T transforms between rectangular coordinate system as a true first order tensor. The quantity T is of the axial vector type.

Just as a contravariant pseudo-vector was obtained from a doubly covariant anti-symmetric tensor, so also can a covariant pseudo-vector similarly be obtained from a doubly contravariant anti-symmetric tensor. The result, comparable to equation (6.8), that would be obtained is

$$t^{ij} = \left(\frac{\partial \bar{x}^1}{\partial x^k}\bar{T}_1 + \frac{\partial \bar{x}^2}{\partial x^k}\bar{T}_2 + \frac{\partial \bar{x}^3}{\partial x^k}\bar{T}_3\right) = T_k, \qquad (6.9)$$

a vector capacity.

2. Vector and scalar products in vector analysis

An important operation in vector analysis is that of the vector product. In tensor analysis the product of two contravariant vectors is given by

$$u^i\, v^j, \qquad (6.10)$$

a second order tensor. But, in order to form a completely anti-symmetric tensor, $u^j v^i$ must be subtracted from equation (6.10). The vector product of tensor analysis corresponding to that of vector analysis will then be:

$$w^{ij} = u^i v^j - v^i u^j, \qquad (6.11)$$

where w^{ij} is a completely anti-symmetric tensor. By using the principles of equation (6.9), the vector product becomes

$$w^{ij} = \Delta\left(\frac{\partial \bar{x}^1}{\partial x^k}\ \bar{W}_1 + \frac{\partial \bar{x}^2}{\partial x^k}\ \bar{W}_2 + \frac{\partial \bar{x}^3}{\partial x^k}\ \bar{W}_3\right) = W_k, \qquad (6.12)$$

a vector density. As before, if this expression is limited to right-handed Cartesian reference frames in three-dimensional space, then $\Delta = +1$. In the symbolism of vector analysis using the unit vectors i, j, k, along axes x, y, z, respectively, the vector product of two vectors u and v can be written in the form

$$W = \begin{vmatrix} i & j & k \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix} = i(u^2 v^3 - u^3 v^2) + j(u^3 v^1 - u^1 v^3) + k(u^1 v^2 - u^2 v^1). \qquad (6.13)$$

It can be remarked that the principles involved in this section can be extended to include the vector products of covariant vectors by using their associated contravariant vectors.

The scalar product of two vectors has been defined [Eq. (4.19)]; it is, however, numerically equivalent for Cartesian reference frames to a contraction of the tensor product of two vectors. The scalar product is given by

$$(u.v) = \sqrt{g_{ij}\, u^i\, u^j}\ .\ \sqrt{g_{ij}\, v^i\, v^j}\ .\ \cos(uv),$$

$$= g_{ij}\, u^i\, v^j = u^i\, v_i. \qquad (6.14)$$

Since

$$g_{ij} = \delta_{ij} \quad \begin{array}{l} = 1 \text{ if } i = j \\ = 0 \text{ if } i \neq j \end{array}$$

the numerical values of u^i and u_i are indistinguishable, and similarly for v^i and v_i; therefore the scalar product is expressible in any of the following equivalent forms:

$$u^i\, v_i = u^i\, v^i = u_i\, v_i = u_i\, v^i. \qquad (6.15)$$

These relations merely demonstrate that the covariant and contravariant components of a vector with respect to Cartesian reference frames are indistinguishable and interchangeable.

3. <u>Stoke's theorem in vector-analysis form</u>

The principles displayed by equations (6.8) and (6.9) have a very important application in the representation of Stoke's theorem. Previously, it was shown that the product of a tensor density and a tensor capacity yielded a true tensor. If a surface element ds^{ij} is represented by a vector capacity, and if the rotation (or curl) r_{ij} is represented as a vector density, then their product can maintain its previous tensor character. Placing

$$\frac{1}{2}\, ds^{ij} = d\sigma_k, \qquad (6.16)$$

where

$$d\sigma_k = \left(\frac{\partial\bar{x}^1}{\partial x^k}d\bar{\sigma}_1 + \frac{\partial\bar{x}^2}{\partial x^k}d\bar{\sigma}_2 + \frac{\partial\bar{x}^3}{\partial x^k}d\bar{\sigma}_3\right) . \tag{6.17}$$

and

$$r_{ij} = \frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i} = R^k, \tag{6.18}$$

where

$$R^k = \Delta^{-1}\left(\frac{\partial x^k}{\partial\bar{x}^1}\bar{R}^1 + \frac{\partial x^k}{\partial\bar{x}^2}\bar{R}^2 + \frac{\partial x^k}{\partial\bar{x}^3}\bar{R}^3\right) , \tag{6.19}$$

then by substitution in equation (3.36) the vector-analysis form of Stoke's theorem is obtained.

$$\oint_c f_k\, dx^k = \int_s R^k\, d\sigma_k . \tag{6.20}$$

The most usual form of Stoke's theorem involves the definition of the rotation (or curl) in terms of the symbolic operator ∇ . The components of equation (6.18) are considered as defining the pseudo-vector R. By associating the unit vectors i, j, k with the axes x, y, z, respectively, it is clear that in the vector-analysis symbolism

$$R = \nabla x f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_2 \end{vmatrix} = i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + j\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \tag{6.21}$$

Introducing equatión (6.21) into equation (6.20), there obtains

$$\oint_c f \cdot dx = \int_s \nabla x f \cdot d\sigma \tag{6.22}$$

where

$$f \cdot dx = f_1\, dx + f_2\, dy + f_3\, dz, \tag{6.23}$$

and

$$(\nabla \times f) \quad d\sigma = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) d\sigma_1 + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) d\sigma_2 + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) d\sigma_3, \tag{6.24}$$

the dot between the vectors standing for a scalar product. Equation (6.22) represents the symbolic vector-analysis form of Stoke's theorem.

4. <u>The gradient, curl, divergence, and Laplacian in vector-analysis forms</u>

Previously [Eq. (2.13)], the gradient of a scalar function was defined as $\frac{\partial \varphi}{\partial x^i}$, where φ was a scalar point function; it was shown to be a typical covariant vector. The components of the gradient in three-dimensional space can be multiplied by their corresponding unit vectors and represented by

$$\text{Grad } \varphi = i \frac{\partial \varphi}{\partial x} + j \frac{\partial \varphi}{\partial x^2} + k \frac{\partial \varphi}{\partial x^3} \tag{6.25}$$

But, symbolically, this is equal to $\nabla \varphi$; therefore

$$\text{Grad } \varphi = \nabla \varphi = i \frac{\partial \varphi}{\partial x} + j \frac{\partial \varphi}{\partial y} + k \frac{\partial \varphi}{\partial z}. \tag{6.26}$$

The absolute <u>divergence</u> is a true scalar and is defined by [Eq. (4.42)]

$$\text{Div } v = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g} \quad v^k \right), \tag{6.27}$$

where v is a contravariant vector. In vector analysis form for three-dimensional space with Cartesian reference frames,

$$\sqrt{g} = \frac{1}{\sqrt{g}} = 1. \qquad (6.28)$$

Thus equation (6.27) can be written as

$$\text{Div } v = \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v^3}{\partial z}, \qquad (6.29)$$

which, in terms of the operator , is equivalent to

$$\text{Div } v = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(i\, v^1 + j\, v^2 + k\, v^3\right),$$

$$= \frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial v^3}{\partial z}. \qquad (6.30)$$

where the dot stands for a scalar product.

The Laplacian of a scalar function was given as [Eq. (4.46)]

$$\nabla^2\varphi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}\; g^{km}\frac{\partial\varphi}{\partial x^m}\right). \qquad (6.31)$$

This function also takes a special form in three-dimensional space with Cartesian reference frames. By using equations (6.28) and

$$g^{km} = \delta^{km} \quad \begin{aligned} &= 1 \text{ if } k = m \\ &= 0 \text{ if } k \neq m \end{aligned} \qquad (6.32)$$

equation (4.29) simplifies to

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}. \qquad (6.33)$$

Sometimes in vector analysis it is desirable to use other than Cartesian reference frames. In this case it is necessary to use formulas (6.27) and (6.31) for the divergence and Laplacian, respectively.

In terms of the unities e_i along the three axes, the fundamental quadratic form becomes

$$ds^2 = g_{ij}\, dx^i\, dx^i = \sum_{ij} e_i\, dx^i\, e_j\, dx^j \cos\theta_{ij}, \qquad (6.34)$$

where θ_{ij} is the angle between the i axis and the j axis. It necessarily follows from equation (6.34) that

$$g_{ij} = e_i\, e_j \cos\theta_{ij}, \qquad (6.35)$$

and

$$g_{ii} = e_i^2 \,. \qquad (6.36)$$

Equation (6.34), can be expressed, therefore, as

$$ds^2 = g_{ij}\, dx^i\, dx^i = \sqrt{g_{ii}} \cdot \sqrt{g_{jj}} \cdot \cos\theta_{ij} \cdot dx^i\, dx^j \quad (6.37)$$

In the special case of orthogonal coordinates equation (6.37) reduces to the following simple form

$$ds^2 = \quad g_{ii}\, (dx^i)^2. \qquad (6.38)$$

In cylindrical coordinates, for example,

$$ds^2 = dr^2 + r^2\, d\theta^2 + dz^2, \qquad (6.39)$$

where

$$g_{ii} = 1; \quad g_{ii} = r^2; \quad g_{33} = 1;$$

$$e_1 = 1; \quad e_2 = r; \quad e_3 = 1. \qquad (6.40)$$

For three dimensional space with orthogonal curvilinear coordinates, the gradient, divergence, and Laplacian assume special and more familiar forms in terms of the e_i. These special forms easily follow from the principles just elicited. From what has preceded it is clear that

$$e_i = \sqrt{g_{ii}} = \frac{1}{\sqrt{g^{ii}}} \tag{6.41}$$

and that the g_{ij} $(i \neq j)$ are zero for orthogonal coordinates. Any vector has either covariant or contravariant components expressible in terms of the e_i. By equations (6.41) and (4.14) it follows that

$$u_i = g_{ij}\, u^j = e_j^2 \;\; u^j, \tag{6.42}$$

or

$$u^i = (e_j^{-1})^2 \; u_i. \tag{6.43}$$

From these relations the magnitude of a vector for orthogonal coordinates is

$$u^2 = g_{ij}\, u^i\, u^j = g_{ii}\, (u^i)^2 = g^{ii}\, (u_i)^2$$

$$= (e_i u^i)^2 = (e_i^{-1}\, u_i)^2 \tag{6.44}$$

The last two forms of equation (6.44) suggest defining the components of a vector in the form

$$U_i = e_i\, u^i = e_i^{-1}\, u_i = \sqrt{u_i\, u^i} \tag{6.45}$$

Then the length of the vector is merely equal to the sum of the squares of the u^i.

The convention of equation (6.45) leads to the following representation of the gradient of a scalar function:

$$(\text{grad } \varphi)_i = \frac{1}{e_i}\frac{\partial \varphi}{\partial x^i} \qquad \text{(No summation)} \tag{6.46}$$

This is the usual vector-analysis form of the gradient for orthogonal curvilinear coordinates in three-dimensional space. Similarly the curl of a vector for orthogonal curvilinear coordinates in three-dimensional space is expressible in the following manner:

$$(\text{curl } f)_k = \frac{F_{ij}}{e_i e_j} = \frac{1}{e_i e_j}\left(\frac{\partial}{\partial x^i}(e_j\, e_j^{-1}\, f_j) - \frac{\partial}{\partial x^j}(e_i\, e_i^{-1}\, f_i)\right)$$

$$= \frac{1}{e_i e_j}\left(\frac{\partial}{\partial x^i}(e_j\, F_j) - \frac{\partial}{\partial x^j}(e_i\, F_i)\right), \qquad (6.47)$$

where $(\text{curl } f)_k$ represents the k component of curl f and i, j, k are permutations of 1, 2, 3.

The divergence of a vector (actually a pseudo-vector) can also be defined in the special terms. Referring again to equation (6.27) and using the relations for orthogonal curvilinear coordinates that

$$g_{ij} = \begin{vmatrix} e_1^2 & 0 & 0 \\ 0 & e_2^2 & 0 \\ 0 & 0 & e_3^2 \end{vmatrix} = (e_1\, e_2\, e_3)^2, \qquad (6.48)$$

and that

$$g^{ij} = \begin{vmatrix} e_1^{-2} & 0 & 0 \\ 0 & e_2^{-2} & 0 \\ 0 & 0 & e_3^{-2} \end{vmatrix} = (e_1\, e_2\, e_3)^{-2}, \qquad (6.49)$$

the divergence becomes

$$\text{Div } v = \frac{1}{e_1\, e_2\, e_3} \; \frac{\partial}{\partial x^k}(e_1\, e_2\, e_3\, v^k) \qquad (6.50)$$

To utilize the special components of equation (6.45), it is necessary to multiply and divide v^k by e_k, and substitute V^k for $e_k\, v^k$. Then equation (6.50) becomes

$$\text{Div } v = \frac{1}{e_1\, e_2\, e_3} \left(\frac{\partial}{\partial x^k} \quad \frac{e_1\, e_2\, e_3}{e_k}\, v^k \right) . \tag{6.51}$$

Equation (6.51) represents the usual vector-analysis form for orthongonal curvilinear coordinates in three-dimensional space.

The Laplacian in the usual vector-analysis form for orthogonal curvilinear coordinates is easily obtained by the substitution of the special relations of equations (6.43), (6.45), and (6.48) into equation (6.31); the result is that

$$\nabla^2 \varphi = \frac{1}{e_1\, e_2\, e_3} \frac{\partial}{\partial x^k} \left(\frac{e_1\, e_2\, e_3}{e_k^2} \frac{\partial \varphi}{\partial x^k} \right) . \tag{6.52}$$

The last few sections give the basic differences between the usual vector analysis and tensor analysis. Further relationships can be developed by employing the ideas involved in these sections, but these examples suffice to indicate the method of approach.

VII. TENSORIAL DYNAMICS

Early methods of approach to dynamical problems depended upon postulates involving vector quantities in Euclidean space. Such methods made possible the logical development of dynamical theories without introducing the concept of energy, but usually required elaborate physical and mathematical interpretation. With the introduction of the scalar concept energy ("vis viva"), the question of whether a dynamical theory involving derivatives of the energy could be devised arose. In 1788, Lagrange introduced in his famous "Mecanique Analytique" (11) a method which resulted in an affirmative answer to the question. Lagrange essentially reduced dynamics to a problem in algebraic analysis.

The method of procedure to be used in this chapter is based upon the postulate:

$$\frac{D}{Dt}\left(\bar{g}_{kk}\frac{d\bar{x}^k}{dt}\right) = \bar{X}_k \quad (k = 1, 2, \ldots, n), \tag{7.1}$$

where D/Dt represents the intrinsic derivative with respect to the parameter t, X_k is the k component of the resultant of the external forces on the system, g_{kk} is the inertial coefficient m_k of the kth particle of the system, and n is equal to three times the number of particles in the system. This fundamental postulate is the tensor form of Newton's second law of motion; therefore it can be transformed to any other coordinates; it is assumed to represent the equations of motion of either N particles in three-dimensional Euclidean space or of one particle in

3N-dimensional ($n = 3N$) Euclidean space. From this fundamental postulate various relations and principles of analytical dynamics will be obtained on succeeding pages.

1. Lagrange's equations of motion

Lagrange's equations of motion are quite easily obtained from the fundamental postulate [Eq. (7.1)] by introducing the concept of kinetic energy. The kinetic energy T of a system of n particles is defined by

$$T = \frac{1}{2} m_k \left(\frac{dx^k}{dt}\right)^2 = \frac{1}{2} \; g_{kk} \left(\frac{dx^k}{dt}\right)^2 . \tag{7.2}$$

The derivative $\frac{dx^k}{dt}$, expressed in terms of T, is given by the equation

$$\frac{\partial T}{\partial \dot{\bar{x}}} = \bar{g}_{kk} \dot{\bar{x}}^k \qquad \left(\dot{\bar{x}}^k = \frac{d\bar{x}^k}{dt}\right). \tag{7.3}$$

Now if equation (7.3) is substituted in (7.1), then the equation

$$\frac{D}{Dt}\left(\frac{\partial T}{\partial \dot{\bar{x}}^k}\right) = \bar{X}_k \tag{7.4}$$

is obtained. When this equation is expanded by using equation (3.32), the expression

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{x}}^k}\right) - \bar{\Gamma}^l_{km} \frac{\partial T}{\partial \dot{\bar{x}}^l} \dot{\bar{x}}^m = \bar{X}_k \tag{7.5}$$

results. The last term on the left in equation (7.5) can be changed to

$$\bar{\Gamma}^l_{km} \frac{\partial T}{\partial \dot{\bar{x}}^l} \dot{\bar{x}}^m = \bar{\Gamma}^l_{km} \frac{\partial T}{\partial \dot{\bar{x}}^l} \bar{g}^{hl} \; \bar{g}_{hl} \dot{\bar{x}}^m$$

$$= \bar{\Gamma}_{h,km} \frac{\partial T}{\partial \dot{\bar{x}}^l} \bar{g}^{hl} \; \dot{\bar{x}}^m . \tag{7.6}$$

But, since

$$\dot{\bar{x}}^m = \frac{\partial T}{\partial \dot{\bar{x}}^\alpha}\,\bar{g}^{m\alpha}\,, \tag{7.7}$$

equation (7.6) can be written in the form

$$\bar{\Gamma}^l_{km}\,\frac{\partial T}{\partial \dot{\bar{x}}^l}\,\dot{\bar{x}}^m = \bar{\Gamma}_{h,km}\left(\frac{\partial T}{\partial \dot{\bar{x}}^l}\,\bar{g}^{hl}\right)\left(\frac{\partial T}{\partial \dot{\bar{x}}}\,\bar{g}^m\right)$$

$$= \frac{1}{2}\left(\bar{\Gamma}_{h,km} + \bar{\Gamma}_{m,kh}\right)\left(\frac{\partial T}{\partial \dot{\bar{x}}^l}\,\bar{g}^{hl}\right)\left(\frac{\partial T}{\partial \dot{\bar{x}}^\alpha}\,\bar{g}^{m\alpha}\right) \tag{7.8}$$

By equation (4.59)

$$\frac{1}{2}\left(\bar{\Gamma}_{h,km} + \bar{\Gamma}_{m,kh}\right) = \frac{\partial \bar{g}_{mh}}{\partial \bar{x}^k}\,;$$

therefore equation (7.8) can assume the form

$$\bar{\Gamma}^l_{km}\,\frac{\partial T}{\partial \dot{\bar{x}}^l}\,\dot{\bar{x}}^m = \frac{1}{2}\,\frac{\partial \bar{g}_{mh}}{\partial \bar{x}^k}\left(\frac{\partial T}{\partial \dot{\bar{x}}^l}\,g^{hl}\right)\left(\frac{\partial T}{\partial \dot{\bar{x}}^\alpha}\,g^{m\alpha}\right) \tag{7.10}$$

$$= \frac{\partial T}{\partial \bar{x}^k}\,.$$

Substituting equations (7.10) into (7.5), the equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{x}}^k}\right) - \frac{\partial T}{\partial \bar{x}^k} = \bar{X}_k \tag{7.11}$$

are obtained. These equations are known as <u>Lagrange's equations of motion</u>. In some other coordinate system x^i, Lagrange's equations transform as follows:

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\bar{x}}^k}\right) - \frac{\partial T}{\partial \bar{x}^k}\right]\frac{\partial \bar{x}^k}{\partial x^i} = \bar{X}_k\,\frac{\partial \bar{x}^k}{\partial x^i}\,, \tag{7.12}$$

or, by equations (2.14) and (4.79),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = X_i\,, \tag{7.13}$$

where

$$T = \frac{1}{2}\ \bar{g}_{kk}\ (\dot{\bar{x}}^{k})^{2} = \frac{1}{2}\ g_{ij}\ \dot{x}^{i}\ \dot{x}^{j}. \tag{7.14}$$

Lagrange's equations represent the motion of N particles in three-dimensional space, but it will be shown later that they also may represent the performance of an electrical circuit with n independent meshes or of an electro-mechanical system, such as a rotating electrical machine, with n degrees of freedom. Geometrically, however, Lagrange's equations represent the motion of a free point in n dimensional metric space; the inertial coefficients forming the components of the metric tensor. In some cases the particle may not be free, but may be constrained to a certain surface; these constrained motions will be discussed in the next section.

2. Constrained motion

So far in this treatment of tensors all transformations have been from one n-dimensional coordinate system to another; the transformation of Lagrange's equations [Eq. (7.11) and Eq. (7.13)] is an example. Geometrically, Lagrange's equations as given represent the free motion of a particle in an n-dimensional space. Sometimes, however, a particle is not completely free but is constrained to move on certain surfaces or in sub-spaces of the n-dimensional space; such motions are called constrained motions. The equations of constraint are mathematical equations stating relationships between the coordinates or their differentials in such a way as to limit the motions to motions consistent with the constraints -- for each reduction by one of the number of coordinates, there is an equation of constraint. Equations of constraint may contain time as a variable. Two cases should be considered. First, when the equations of constraint are expressed in terms of the coordinate variables or in terms of differentials of the coordinate variables by equations which are exact differentials, the resulting dynamical system is said to be holonomous; therefore the equations of constraint represent the use of too many variables. The second possibility is that non-integrable equations of constraint in terms of the differentials of the coordinates exist so that corresponding relations between the coordinates cannot be obtained. When this is the case, the constrained dynamical system is said to be non-holonomous. These two cases can be represented in tensor language.

The reduction of the number of coordinates in a holonomic system can be expressed by a rectangular transformation representing a transformation from n old coordinates x^i to n new coordinates q^i with the last r of the coordinates q^i null. The r holonomous equations of constraint $(r < n)$ can be expressed by the r equations (of rank r)

$$F_j\,(x^i, t) = 0 \tag{7.15}$$

These equations can be solved for any $(n - r)$ independent variables, which may represent new coordinates. If the last r of the new coordinates q^i are assumed null, then

$$\begin{aligned} q^{n-r+1} &= F_1(x^i, t) = 0; \\ q^{n-r+2} &= F_2(x^i, t) = 0; \\ &\cdots\cdots\cdots\cdots \\ q^{n} &= F_r(x^i, t) = 0. \end{aligned} \tag{7.16}$$

These equations $\left[\text{Eq. } (7.16)\right]$ can be solved for the x^i in terms of the q^i and t; the solution yields the n equations

$$x^i = x^i(q^1, \ldots, q^{n-r}, t), \tag{7.17}$$

from which the rectangular transformation matrix $\left[\frac{\partial x^i}{\partial q^j}\right]$ is obtained. The determinant of the transformation matrix $\left[\frac{\partial x^i}{\partial q^j}\right]$ is singular; therefore the inverse transformation matrix $\left[\frac{\partial q^j}{\partial x^i}\right]$ cannot be unique. Since the elements $\frac{\partial q^j}{\partial x^i}$ are not defined, transformations of contravariant tensors are not possible. If a metric is defined for the n-space, the contravariant indices of a tensor may be changed, however, to covariant indices and the transformations performed.

Non-holonomous equations of constraint are functions in general of the n coordinate variables and either the n differential of the coordinate variables or the intrinsic derivatives of the n coordinate variables. If r non-holonomous equations of constraint exist, they may be expressed in the form

$$f_j(x^i, dx^i, t) = 0 \qquad (j = 1, \ldots, r). \tag{7.18}$$

These equations may be solved for any $(n - r)$ independent differentials which may represent the differentials of arc, ds^i, of an ennuple of congruences. If the last r of the n differentials ds^i are assumed null, then

$$\begin{aligned} ds^{n-r+1} &= f_1(x^i, dx^i, dt) = 0; \\ ds^{n-r+2} &= f_2(x^i, dx^i, dt) = 0; \\ &\cdots\cdots\cdots\cdots\cdots \\ ds^n &= f_r(x^i, dx^i, dt) = 0. \end{aligned} \tag{7.19}$$

When equations (7.19) are solved for the n old differentials, the n equations

$$dx^i = \phi^i(ds^1, \ldots, ds^{n-r}, x^i, t) \tag{7.20}$$

are obtained. As a special equation (7.20) may assume the form

$$dx^i = a_k{}^i\, ds^k + T\, dt \quad (k = 1, \ldots, r) \tag{7.21}$$

where the coefficients $a_k{}^i$ and T may be functions of the variables x^i and the parameter t. Non-holonomic constraints will be given further consideration in a succeeding section.

3. Lagrange's equations for Holonomic systems

A transformation of Lagrange's equations [Eq. (7.13)] from an n-space to a subspace can easily be accomplished. Using the transformation matrix $\left[\dfrac{\partial x^i}{\partial q^j}\right]$, the transformation is given by the (n - r) equations

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i}\right]\frac{\partial x^i}{\partial q^j} = X_i\,\frac{\partial x^i}{\partial q^j} \quad (i = 1, \ldots, n) \tag{7.22}$$

or

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{q}^j}\right) - \frac{\partial T'}{\partial q^j} = Q_j \tag{7.23}$$

where

$$Q_j = \overline{X}_i\,\frac{\partial x^i}{\partial q^j}\,. \tag{7.24}$$

It should be carefully noted that in equations (7.23) that $\frac{d}{dt}$ stands for the time derivative following the motion; that is,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial q^i} \dot{q}^i + \frac{\partial}{\partial \dot{q}^i} \ddot{q}^i \tag{7.25}$$

Equations (7.23) are Lagrange's equations for a dynamical system with holonomous constraints; they represent the motion of a particle in an (n - r) dimensional subspace.

Lagrange's equations for holonomic system can be expanded by substituting for the kinetic energy T' and performing the indicated operations. Using

$$T' = \frac{1}{2} g_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta \tag{7.26}$$

in equation (7.23), there obtains

$$2 g_{ij} \ddot{q}^i + \left(\frac{\partial g_{\alpha j}}{\partial q^\beta} - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial q^j} \right) \dot{q}^\alpha \dot{q}^\beta = Q_j, \tag{7.27}$$

where the constraints depend explicitly upon time. If the equations of constraint do not contain time explicity "cyclic" variable then the partial derivatives with respect to time [Eq. (7.25)] are zero; that is,

$$\frac{d}{dt} = \frac{\partial}{\partial q^i} \dot{q}^i + \frac{\partial}{\partial \dot{q}^i} \ddot{q}^i. \tag{7.28}$$

In this case equation (7.27) becomes

$$g_{ij} \ddot{q}^i + \left(\frac{\partial g_{\alpha j}}{\partial q^\beta} - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial q^j} \right) \dot{q}^\alpha \dot{q}^\beta = Q_j, \tag{7.29}$$

which is equivalent to

$$g_{ij} \ddot{q}^i + \Gamma_{j,\alpha\beta} \dot{q}^\alpha \dot{q}^\beta = Q_j. \tag{7.30}$$

where $\Gamma_{j,\alpha\beta}$ is the Christoffel symbol of the first kind [Eq. (4.60)]. These expanded forms become very useful in practical application.

4. Transformation of Lagrange's equations to an ennuple of congruences

Lagrange's equations of motion can be expressed with reference to an ennuple of congruences instead of true coordinates. To accomplish such a transformation the principles outlined in chapter V will be used.

Lagrange's equations for true coordinates are given by equations (7.13). By a transformation to an ennuple of congruences they become

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i}\right] \frac{\partial x^i}{\partial s^j} = X_i \frac{\partial x^i}{\partial s^j} . \tag{7.31}$$

This form is not as convenient as a form utilizing the kinetic energy expressed in terms of the new differentials ds^j and the old variables x^i, and the derivatives expressed as directional derivatives. It should be noted that a transformation of the type used in equation (7.31) is a transformation between differentials of arc as the underlying variables may be undefined. In terms of the n new differentials and the n old variables the kinetic energy can be expressed by

$$\overline{T} = \frac{1}{2} \, \overline{g}_{\alpha\beta} (x^k, ds^a) \frac{ds^\alpha}{dt} \frac{ds^\beta}{dt} \tag{7.32}$$

By using the principles of directional differentiation of chapter V, the quantity

$$\frac{\partial T}{\partial \dot{x}^i} = \frac{\partial \overline{T}}{\partial \dot{s}^\alpha} \frac{\partial s^\alpha}{\partial x^i} ; \tag{7.33}$$

$$\frac{d}{dt} \frac{\partial T}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \dot{s}^\alpha} \frac{\partial s^\alpha}{\partial x^i}\right)$$

$$= \frac{\partial s}{\partial x^i}\left(\frac{d}{dt} \frac{\partial \overline{T}}{\partial \dot{s}^\alpha}\right) + \frac{\partial \overline{T}}{\partial \dot{s}^\alpha}\left(\frac{\partial}{\partial x^m} \frac{\partial s^\alpha}{\partial x^i}\right) \frac{\partial x^m}{\partial s^l} \dot{s}^l ; \tag{7.34}$$

and

$$\frac{\partial T}{\partial x^i} = \frac{\partial \overline{T}}{\partial x^i} + \frac{\partial \overline{T}}{\partial \dot{s}^\beta} \frac{\partial \dot{s}^\beta}{\partial x^i}$$

$$= \frac{\partial \overline{T}}{\partial x^i} + \frac{\partial \overline{T}}{\partial \dot{s}^\beta} \left(\frac{\partial}{\partial x^i} \frac{\partial s^\beta}{\partial x^m} \right) \frac{\partial x^m}{\partial s^l} \dot{s}^l. \qquad (7.35)$$

When equations (7.34) and (7.35) are substituted in equation (7.31), and when some dummy indices are changes, the equations

$$\frac{d}{dt}\left(\frac{\partial \overline{T}}{\partial \dot{s}^j}\right) - \frac{\partial \overline{T}}{\partial s^j} + \frac{\partial \overline{T}}{\partial \dot{s}^\alpha} \left(\frac{\partial}{\partial x^m} \frac{\partial s^\alpha}{\partial x^i} - \frac{\partial}{\partial x^i} \frac{\partial s}{\partial x^m} \right)$$

$$\frac{\partial x^m}{\partial s^l} \frac{\partial x^i}{\partial s^j} \dot{s}^l = X_i \frac{\partial x^i}{\partial s^j} \qquad (7.36)$$

are obtained. These equations of motion are represented in terms of an ennuple of congruences. This last form of the equations of motion is equivalent to a form given by Whittaker (26, p. 43).

5. The equations of motion for non-holonomic systems

In the case of holonomic dynamical systems the number of independent coordinates required to specify the configuration of a system at any time is equal to the number of degrees of freedom of the system. For non-holonomic systems, however, the number of independent coordinates needed to specify the configuration at any time may be greater than the number of degrees of freedom, owing to the fact that the system is subjected to constraints which are supposed to do no work, and which are expressed by a number of non-integrable expressions of the form

$$X^i{}_j \, dx^j + T^i \, dt = 0 \quad (i = r + 1, \ldots, n; \; 1 \leq r < n), \qquad (7.37)$$

where the $\overline{A}^i{}_j$'s and the T^i's are functions of the coordinates x^i and the t. Such a system of (n - r) equations, if the system has no integrals whatsoever, and if the terms in dt are neglected, is said to represent a single r-dimensional non-holonomic space in an n-space (23, p. 12).

A non-holonomic system may be regarded as subject to the conditions of the form expressed by equations (7.37), or it may be regarded as a system acted upon by certain additional external forces; namely, the forces which must be exerted by the constraints in order to compel the system to fulfill the conditions. By assuming the latter point of view, and by letting $X'_k \ \delta x^i$ be the work done on the system by the additional forces in a displacement δx subject to the conditions of constraint, and by letting $X_i \ \delta x^i$ be the corresponding work for the original external forces, Lagrange's equations can be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = X_i + X'_i. \tag{7.38}$$

Even though the forces X'_i are unknown, they are such that in a displacement consistent with the restraints they do no work. For all values of the ratios of differentials which satisfy equations (7.37), it follows that the work done by the restraining forces must be zero; that is, the relation

$$X'_i \ \delta x^i = \rho^j \ \overline{A}^j{}_i \ \delta x^i, \tag{7.39}$$

where the ρ^j's are undetermined multipliers, must be satisfied. The equations of motion of the system are now represented by the (n + r) equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = X_i + \rho^j \bar{A}^j{}_i \quad (j = 1, \ldots, r)$$

$$\bar{A}^j{}_\alpha \dot{x}^\alpha + T^j = 0 \tag{7.40}$$

where the n + r unknown quantities are $x^1, x^2, \ldots, x^n$ and $\rho^1, \rho^2, \ldots, \rho^r$. The solutions of these equations represent the performance of the system.

An interesting point of view obtains when the multipliers p^i are considered as suitable factors to make the field of contravariant vectors $\bar{a}_i{}^j$ conjugate to the covariant vectors $\bar{a}^i{}_j = \rho^i \bar{A}^i{}_j$ consist of unit vectors. Thereby, an ennuple of congruences is obtained with reference to which equations (7.37) take the form

$$ds^i = \bar{a}^i{}_j \, dx^j = 0 \qquad (j = 1, \ldots, n), \tag{7.41}$$

when the T^i's are zero; the remaining n - r differentials of arc do not vanish. A function ϕ is an integral of equation (7.41) if $\frac{\partial \phi}{\partial s^i} = 0$, for $i = 1, 2, \ldots, r$; that is, if and only if the family of hypersurfaces ϕ = constant contains the first r congruences of the ennuple. Then the systems of equations to be satisfied is given by following the procedure used for equations (7.36),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}^j}\right) - \frac{\partial T}{\partial s^j} + \frac{\partial T}{\partial \dot{s}^\alpha}\left(\frac{\partial \bar{a}^\alpha{}_i}{\partial x^m} - \frac{\partial \bar{a}^\alpha{}_m}{\partial x^i}\right)\bar{a}_j{}^i \, \bar{a}_l{}^m \, \dot{s}^l = Q_j \tag{7.42}$$

and

$$ds^j = \bar{a}^j{}_\alpha \, dx^\alpha ,$$

where the last r of the ds^i are zero due to orthogonality between the last r components of $\bar{a}$ and dx.

If equation (7.42) is expanded by substituting

$$T = \frac{1}{2}\ g_{\alpha\beta}\,\dot{s}^{\alpha}\ \dot{s}^{\beta}\ ,$$

and by performing the indicated differentiations, the following expression for the equations of motion is obtained:

$$g_{\alpha j}\ \ddot{s}^{\alpha} + \left[\left(\frac{\partial g_{\alpha j}}{\partial s^{\beta}} - \frac{1}{2}\ \frac{\partial g_{\alpha\beta}}{s^{j}}\right) + g_{1\beta}\left(\frac{\partial \overline{a}^{-1}_{i}}{\partial x^{m}} - \frac{\partial \overline{a}^{-1}_{m}}{\partial x^{i}}\right)\right.$$

$$\left.\overline{a}_{j}^{\ i}\ \overline{a}_{\alpha}^{\ m}\right]\dot{s}^{\alpha}\ \dot{s}^{\beta} = Q_{j}. \qquad (7.43)$$

This expression is another form of the equations of motion of a non-holonomic dynamical system.

By returning to the fundamental postulate [Eq. (7.1)] and by transforming it to an ennuple of congruences, the following equation is obtained:

$$\frac{D}{Dt}\left(\overline{g}_{kk}\ \frac{d\overline{x}^{k}}{dt}\right)\frac{\partial \overline{x}^{k}}{\partial s^{i}} = \overline{X}_{k}\ \ \frac{\partial \overline{x}^{k}}{\partial s^{i}}\ . \qquad (7.44)$$

When this expression is expanded, it takes the form

$$g_{ij}\ \frac{d^{2}s^{j}}{dt^{2}} + L_{i,\alpha\beta}\ \ \frac{ds^{\alpha}}{dt}\ \frac{ds^{\beta}}{dt} = X_{i}, \qquad (7.45)$$

where

$$L_{i,\alpha\beta} = \Gamma_{i,\alpha\beta} - S_{i,\alpha\beta}\ ,$$

$$= \frac{1}{2}\left(\frac{\partial g_{\alpha i}}{\partial s^{\beta}} + \frac{g_{\beta i}}{\partial s^{i}} - \frac{\partial g_{\alpha\beta}}{\partial s^{i}}\right) - S_{i,\alpha\beta} \qquad (7.46)$$

is the metric connection; $S_{i,\alpha\beta}$ is the torsion tensor. If a transformation of equation (7.45) to another congruence were made following the method of chapter V, then the resulting equation would have the same form as equation (7.45); therefore the transformation need not be given. The form given by equation (7.43) is the preferable form from the viewpoint of application as less labor is involved in calculating its components than in calculating the components of either the form given by equation (7.42) or that given by equation (7.45).

Some of the principles given in this chapter will be applied to the analysis of a typical electro-mechanical problem, the rotating electrical machine, in the next chapter. The equations of performance of the rotating electrical machine will be transformed from a set of Riemannian coordinates to an ennuple of congruences in order to simplify their solution.

VIII. ROTATING ELECTRICAL MACHINERY

The general outward appearance of one type of rotating electrical machine is so similar to that of another type that it is often very difficult to distinguish between the various types from appearance alone. Even electrical measurements made at the coil terminals of the machine fail to furnish sufficient additional data for the complete identification of the machine. A little experience with electrical machinery soon leads one to the conclusion that the essential difference between the various types of machines is the way in which the coils are interconnected. Kron (9, 10) has based his "representative" machines on this essential difference and has considered the various types of machines as resulting from the method of interconnection of the coils of his representative machines. Kron's mathematical tools were those of tensor analysis; his method of reasoning was that of differential geometry. The author wishes to apply the methods of the intrinsic tensor analysis of an ennuple of congruences to the rotating-machine problem. The equations of motion in tensor form are assumed to represent the performance of a general electro-mechanical system; therefore they will represent a typical electro-mechanical system, the rotating electrical machine. Although no special stress is placed on electro-mechanical equivalents, the recognition of their existence is necessary in order to form this unified theory of electrical and mechanical systems.

An application of the general method to the analysis of an induction motor is given. This application is an attempt to formulate in the language of the intrinsic tensor analysis some of Stanley's results (21), his general method of approach being similar to that used by Park (16) and others.

1. Analytical approach to rotating electrical machines

The differential equations which represent the performance of electrical circuits are analogous in form with those which represent the performance of mechanical circuits; also there is a one-to-one correspondence between the concepts of electrical and of mechanical systems. Knowing the analogous concepts makes it possible for one to solve mechanical systems in terms of equivalent electrical systems, or vice versa. It is also possible to solve interlinked mechanical and electrical systems without reducing the system to either an equivalent electrical or an equivalent mechanical system. This latter method seems preferable to the author; therefore it will be used to solve a typical electro-mechanical system, the rotating electrical machine. Although it is actually an analysis of a particular electro-mechanical system, this treatment should be considered as a treatment of a method of analysis which applies to electrical, mechanical, and electro-mechanical systems as special cases. For reference purposes, some of the formal analogies between electrical and mechanical circuit concepts are given in the following table:

Table I

Electro-Mechanical Equivalents

Linear Motion		Rotational Motion		Electrical Circuit		
Quantity	Unit	Quantity	Unit	Quantity	Unit	Symbol
Force	dyne	Torque	dyne-cm.	Electromotive force	volt	e_i
Displacement	cm	Angular displacement	radian	Charge	coulomb	q^i, x^i, or s^i
Velocity	cms sec	Angular velocity	radian sec	Current	ampere	$\dot{q}^i, \dot{x}^i$, or $\dot{s}^i$
Mass	gram	Moment of inertia	gram cm^2	Inductance	henry	g_{ij}
Linear compliance	cms dyne sec	Rotational compliance	radians dyne - cm	Capacitance	farad	C_{ij}
Mechanical resistance	dyne sec cm	Rotational resistance	dyne cm sec radian	Resistance	ohm	R_{ij}
Power	ergs sec	Power	ergs sec	Power	watts	P

In this table it should be noted that all analogous quantities are represented by the same symbol; they will be treated identically in the mathematics that follows.

For a general electro-mechanical system it is necessary to know the following three sets of design constants in order to perform an analysis of the system: the resistances; the inductances, including the inertial coefficients; and the capacitances, including the compliances. For a rotating machine only two sets of design constants, the inductances and the resistances, usually are required because the capacitances are so small. For convenience, design constants should be arranged in some tabular form; as an example, the inductances are represented in the form

$$g_{ij} = \begin{array}{c|cccc|c} & q^1 & q^2 & \cdots & q^n & \\ \hline i \downarrow & g_{11} & g_{12} & \cdots & g_{1n} & q^1 \\ & g_{21} & g_{22} & \cdots & g_{2n} & q^2 \\ & \cdots & \cdots & \cdots & \cdots & \\ & g_{n1} & g_{n2} & \cdots & g_{nn} & q^n \\ \hline & & j \rightarrow & & & \end{array} \qquad (8.1)$$

Most rotating electrical machines may be grouped into one of the following types: those with slip-rings; those with commutators. An alternator is a typical slip-ring machine; an ordinary shunt-excited d-c dynamo represents a typical commutator machine. Riemannian geometrical methods are suited for the representation of machines of the slip-ring type as transformations of a particular machine, but non-Riemannian geometrical methods are required for the representation of all machines, including both slip-ring and commutator types, as transformations of a particular machine.

Since the commutator type of machine differs from the slip-ring type essentially only in the manner of external connection of its coils, it seems reasonable to believe that one can be considered as a transformation of the other. This is the case, but non-Riemannian methods of analysis are required. In fact, most machines can be reduced, for purposes of analysis, to the commutator type by a suitable transformation of coordinates. The advantage gained by such a transformation is a simplification that permits an easier solution of the equations of performance.

Blondel furnished the clue for the type of transformation needed to reduce slip-ring machines to equivalent commutator machines. In an original paper (1) Blondel suggested that the armature reaction of an alternator is essentially no different from that of a d-c machine. By considering two-pole machines, he demonstrated that shifting the brushes of a d-c machine was analogous to changing the power factor of the load on an alternator. As the armature reaction in a d-c machine had been considered in terms of demagnetizing and cross-magnetizing components, along and perpendicular to, respectively, the line joining the field poles, Blondel was led to consider the armature reaction of an alternator drawing balanced currents to be divided into components along and perpendicular to the line joining the field poles. He also assumed that each component was produced by its own current. This assumption makes it possible to associate a set of coils with each component of current

These coils should be stationary with respect to the field curcuit; also they should be arranged so that their magnetic axes correspond with the direction of the components of armature reaction which they are to represent. Implicitly contained in this type of resolution is the assumption that the resolved components of armature reaction represent the same sort of space-distributions of magnetomotive force. For this treatment the convenient sinusoidal space-distribution is assumed. If Blondel's resolution is expressed mathematically in terms of currents, it can be considered as a linear transformation from one set of currents to another set. But it is known that the differentials of coordinate variables transform in the same way as do currents; therefore Blondel's resolution defines also a transformation from one set of differentials to another set. A closer examination of the transformations involved brings out the fact that if the angular position of the rotor of a machine is considered as a coordinate variable, that the relations between the differentials cannot be integrated to obtain the underlying variables; in other words, an intrinsic transformation is involved, that is, a transformation between a set of coordinates and an ennuple of congruences.

The statements of the previous paragraph can be clarified somewhat by considering a special case. A single phase alternator with a sinusoidal space-distribution of armature and with a stationary salient two-pole field with projection axes along and perpendicular to its magnetic axis ? is assumed; therefore a moving axis can be associated with the direction of the resultant armature reaction. This moving axis will be denoted by q^1, and the stationary axes will be denoted by s^1 and s^2, along and perpendicular to, respectively, the magnetic axis of the field. If the angular position θ of the moving armature is measured between the magnetic axis of the armature coil and the magnetic axis of the field, then the transformations between the time rates of change of the variables and between the differentials of the variables are given by

$$\dot{q}^1 = \cos\theta\, \dot{s}^1 + \sin\theta\, \dot{s}^2,$$

and

$$dq^1 = \cos\theta\, ds^1 + \sin\theta\, ds^2, \tag{8.2}$$

respectively. For these equations to be exact differentials it is necessary that

$$\frac{\partial}{\partial\theta}\frac{\partial q^1}{\partial s^1} = \frac{\partial}{\partial s^1}\frac{\partial q^1}{\partial\theta}$$

As

$$\frac{\partial q^1}{\partial s^1} = \cos\theta$$

and

$$\frac{\partial q^1}{\partial\theta} = 0,$$

it is clear that the equations (8.2) do not represent exact differentials.

A more general Blondel-type of transformation is exemplified by the application of Blondel's principles to the three-phase alternator. As before, sinusoidal space-distributions of the magneto-motive force of armature reaction is assumed. The field circuit is again assumed to be of the salient two-pole type with stationary axes along and perpendicular to its magnetic axis. Further, it is assumed that the angular position of the rotor is measured between the magnetic axis of any one of the phase coils of the armature as a reference and the magnetic axis of the field. The other two phase coils of the armature are spaced at 120 degree intervals from the first phase coil. If the stationary axes are denoted by the variables s^i and the moving axes by the variables q^i, then the transformation from the stationary to the moving axes is given by the equations

$$\begin{aligned} dq^1 &= \cos\theta\, ds^1 + \sin\theta\, ds^2 \\ dq^2 &= \cos(\theta + 120)\, ds^1 + \sin(\theta + 120)\, ds^2 \\ dq^3 &= \cos(\theta - 120)\, ds^1 + \sin(\theta - 120)\, ds^2. \end{aligned} \tag{8.3}$$

In general, however, in case the sum of the three phase-currents is not zero, a third independent stationary coordinate is required. This additional coordinate can be assumed to have equal projections on all the moving axes; in this case equations (8.3) will be replaced by

$$\begin{aligned} dq^1 &= \cos\theta\, ds^1 + \sin\theta\, ds^2 + ds^3 \\ dq^2 &= \cos(\theta + 120)\, ds^1 + \sin(\theta + 120)\, ds^2 + ds^3 \\ dq^3 &= \cos(\theta - 120)\, ds^1 + \sin(\theta - 120)\, ds^2 + ds^3. \end{aligned} \tag{8.4}$$

If equations (8.4) are solved for the ds^i in terms of the dq^i, a transformation of the components dq^i to the components ds^i is obtained. Such a solution yields the following equations

$$\begin{aligned} ds^1 &= \tfrac{2}{3}\cos\theta\, dq^1 + \tfrac{2}{3}\cos(\theta + 120)\, dq^2 + \tfrac{2}{3}\cos(\theta - 120)\, dq^3 \\ ds^2 &= \tfrac{2}{3}\sin\theta\, dq^1 + \tfrac{2}{3}\sin(\theta + 120)\, dq^2 + \tfrac{2}{3}\sin(\theta - 120)\, dq^3 \\ ds^3 &= \tfrac{1}{3}\, dq^1 + \tfrac{1}{3}\, dq^2 + \tfrac{1}{3}\, dq^3. \end{aligned} \tag{8.5}$$

The coefficients in equations (8.5) must be conjugate to those in equations (8.4) because they result from the solution of equations (8.4). In other words, if the coefficients in equations (8.4) are denoted by $\frac{\partial q^i}{\partial s^j}$, and if those in equations (8.5) are denoted by $\frac{\partial s^j}{\partial q^i}$, then the two sets must obey the relationships

$$\frac{\partial q^i}{\partial s^j}\frac{\partial s^k}{\partial q^i} = \delta_j^k ; \quad \frac{\partial s^i}{\partial q^j}\frac{\partial q^k}{\partial s^i} = \delta_j^k . \qquad (8.6)$$

With these final relationships on hand, the method of analyzing general rotating machines can be resumed.

The equations of motion for dynamical systems have been assumed to apply to electro-mechanical systems. If a slip-ring machine (one with moving reference axes) is analyzed as a slip-ring type of machine, that is, if it is not referred to stationary axes but to axes moving with the conductors, then the equations (7.25) represent its performance. It is necessary, however, that the quantities Q_j be defined by

$$Q_j = e_j - R_{ij}\, q^i, \qquad (8.7)$$

where the quantities e_j are either applied torques or applied voltages and the quantities $R_{ij}\, q^i$ are either resistive torques or resistive voltage-drops. By substituting equation (8.7) in (7.25), the following equations representing the performance are obtained:

$$R_{ij}\, \dot{q}^i + g_{ij}\, \ddot{q}^i + \left(\frac{\partial g_{\alpha j}}{\partial q^\beta} - \frac{1}{2}\, \frac{\partial g_{\alpha\beta}}{q^j}\right) \dot{q}^\alpha \;\; \dot{q}^\beta \;\; = e_j . \qquad (8.8)$$

In order to determine the performance of the given slip-ring machine expressed along moving reference axes it becomes necessary to solve a set of equations of the form given by equation (8.8).

For rotating electrical machines, it so happens that the quantities g_{ij} are, in general, not constant but that they depend upon the position of the rotor; therefore the application of expression (8.8) results in a set of differential equations with coefficients which are functions of the rotor position. Such equations are usually difficult to solve; therefore if a transformation which would reduce the coefficients to constants or to simple forms could be found, it is possible that much of the difficulty might be removed. The Blondel type of transformation [Eq. (8.4) and (8.5)] will do just this. It is necessary, however, to transform the quantities in equations (8.8) to equivalent quantities along intrinsic reference frames. In previous chapters, the method of transformation and the resulting unexpanded form of the performance equations were given [Eq. (7.36)] in Lagrangian form. The new performance equations are given by

$$R'_{\alpha k}\, \dot{s}^{\alpha} + g'_{\alpha k}\, \ddot{s}^{\alpha} + \left[\left(\frac{\partial g'_{\alpha k}}{\partial s} - \frac{1}{2}\frac{\partial g'_{\alpha\gamma}}{\partial s^{k}}\right) + g'_{\alpha\beta}\left(\frac{\partial}{\partial q^{m}}\frac{\partial s^{\beta}}{\partial q^{i}} - \frac{\partial}{\partial q^{i}}\frac{\partial s^{\beta}}{\partial q^{m}}\right)\left(\frac{\partial q^{i}}{\partial s^{k}}\;\frac{\partial q^{m}}{\partial s}\right)\right]\dot{s}^{\alpha}\;\dot{s}^{\gamma} = e'_{k} \qquad (8.9)$$

where the primed quantities are the transformed quantities. These equations define, in general, the motion of a point in a non-Riemannian metric space. When they are solved, equations (8.9) represent the performance a machine which is equivalent through a transformation to a given slip-ring machine has; for example, this equivalent machine might be a commutator type as happens to be the case if a Blondel type of transformation is used.

2. Formation of the components of the inductance and the resistance tensors

The inductances, both self and mutual, of the coils of a rotating electrical machine depend, in general, upon the position of the rotor. In order to simplify their mathematical representation as well as the representation of other quantities, it is necessary to introduce some assumptions. The following assumptions are made:

1. Saturation, hysteresis, and eddy currents are negligible.
2. The variation of self- or mutual-inductances with the position of the rotor follows sine curves.
3. The armature is smooth, and has balanced sinusoidally distributed windings.
4. Only two field poles exist for synchronous or for d-c machines.
5. Induction-type machines have a uniform air gap and balanced sinusoidally distributed rotor windings.
6. Resistance changes due to heating are negligible.
7. Three-phase armature circuits are used unless otherwise stated.

The self-inductance of an armature phase is a maximum for a salient two-pole machine when the direct axis is lined up with the magnetic axis of the phase and a minimum where the quadrature axis is in line with it. If the rotor position is measured from the magnetic axis of phase 1, then the self-inductance of phase 1 is given by

$$L_1 = L_a + L_m \cos 2\,\theta, \tag{8.10}$$

where L_a is greater than L_m. This functional representation can be determined either by calculation or test.

Similarly, for phases 2 and 3 the self-inductances are given by

$$L_2 = L_a + L_m \cos 2(\theta - 120), \qquad (8.11)$$

and

$$L_3 = L_a + L_m \cos 2(\theta + 120). \qquad (8.12)$$

The mutual inductances between the armature phase coils vary in a similar way. But before the mutual inductances can be defined it is necessary to define the relative directions of the currents. Positive currents are defined in the same relative direction; that is, positive current in phase 2 tends to produce a flux-linkage in phase 1 in a direction opposite to that which is produced by a positive current in phase 1. Now the mutual inductances between the armature phase coils are given by the following equations:

$$M_{12} = - M_a - M_m \cos 2(\theta - 60), \qquad (8.13)$$
$$= - M_a + M_m \cos (2\theta - 120);$$

$$M_{23} = - M_a + M_m \cos 2\theta; \qquad (8.14)$$
$$M_{13} = - M_a + M_m \cos (2\theta + 120). \qquad (8.15)$$

The maximum values of the variation of one of these mutual inductances occur at double the frequency which corresponds to the rotor speed. A particular variation, that between phase 1 and phase 2 for example, has its maximum values when the magnetic axis of the field bisects the angle between the adjacent coil sides of phases 1 and 2. In practical cases, the variations of the mutual-inductances are almost equal to those of the self-inductances.

The maximum mutual-inductance between any phase of the armature and the main field circuit will exist when the direct axis of the field is in line with the magnetic axis of the particular phase coil. The variations of these mutual-inductances for the three-phases are given by:

$$M_{1f} = M_f \cos \theta; \tag{8.16}$$

$$M_{2f} = M_f \cos(\theta - 120); \tag{8.17}$$

$$M_{3f} = M_f \cos(\theta + 120). \tag{8.18}$$

The self-inductance of the main field circuit is constant and is represented by L_{ff}.

In case additional windings exist on either the stator or rotor of a machine the inductance variations can be represented by equations similar to those just given.

If an induction type of machine is being considered, then both the rotor and the stator are assumed to be smooth and to contain sinusoidally distributed balanced windings; therefore the self-inductances and the mutual-inductances between stator coils or between rotor coils are constant. The mutual inductances between the stator and the rotor coils are the only inductances which vary; they are given by the following equations:

$$M_{14} = M \cos \theta; \tag{8.19}$$

$$M_{15} = M \cos (\theta + 120); \tag{8.20}$$

$$M_{16} = M \cos (\theta - 120); \tag{8.21}$$

$$M_{24} = M \cos (\theta - 120); \tag{8.22}$$

$$M_{25} = M \cos \theta; \tag{8.23}$$

$$M_{26} = M \cos (\theta + 120); \tag{8.24}$$

$$M_{34} = M \cos (\theta + 120); \tag{8.25}$$

$$M_{35} = M \cos (\theta - 120); \tag{8.26}$$

$$M_{36} = M \cos \theta. \tag{8.27}$$

The numbers 1, 2, and 3 represent the stator (armature) coils; the numbers 4, 5, and 6 represent the rotor coils. M is the maximum value of the variation of these mutual inductances. It should be noted that

$$M_{ij} = M_{ji}. \tag{8.28}$$

In forming these variations of stator-to-rotor inductances, a positive current in phase 1, for example, was assumed to produce a flux-linkage in the same direction as a positive current in phase 4.

The value of the moment of inertia of the rotor is also a component of the inductance tensor; it is constant. It will be represented by j_r.

Sufficient information is now on hand so that a typical inductance tensor can be formed. The components of such an inductance tensor should be chosen from among the various equations of this section in such a way as to represent the given machine. Equation (8.1) represents a method of writing the components.

The components of the resistance tensor are easily formed. The mutual-resistances are usually zero in a rotating electrical machine; therefore there remain only the self-resistances of the various coils, and the friction and windage resistance of the rotor. The components of the resistance tensor should be arranged in a form similar to that used for the inductance tensor.

3. The induction motor

The wound-rotor induction motor is quite easily represented by the methods of this chapter. In order to more clearly demonstrate the principles and methods involved, this particular rotating machine will be analyzed in some detail.

The inductance tensor for an induction motor of the wound-rotor type is formed from the components given in the previous section. It is represented by

$$
g_{ij} = \begin{array}{c|ccccccc|c}
 & q^1 & q^2 & q^3 & q^4 & q^5 & q^6 & q^7 & \\
\hline
 & L_a & -M_a & -M_a & M\cos\theta & M\cos(\theta+120) & M\cos(\theta-120) & 0 & q^1 \\
i\downarrow & -M_a & L_a & -M_a & M\cos(\theta-120) & M\cos\theta & M\cos(\theta+120) & 0 & q^2 \\
 & -M_a & -M_a & L_a & M\cos(\theta+120) & M\cos(\theta-120) & M\cos\theta & 0 & q^3 \\
 & M\cos\theta & M(\cos\theta-120) & M\cos(\theta+120) & L_b & -M_b & -M_b & 0 & q^4 \\
 & M\cos(\theta+120) & M\cos\theta & M\cos(\theta-120) & -M_b & L_b & -M_b & 0 & q^5 \\
 & M\cos(\theta-120) & M\cos(\theta+120) & M\cos\theta & -M_b & -M_b & L_b & 0 & q^6 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & J_r & q^7 \\
\hline
 & & & & j \longrightarrow & & & &
\end{array}
\qquad (8.29)
$$

where the variables q^1, q^2, and q^3 are stator variables corresponding to phases 1, 2, and 3, respectively; and q^4, q^5, and q^6 are the rotor variables corresponding to phases 4, 5, and 6, respectively; also q^7 is equal to θ, the angular position of the ratio. The mutual-inductances between stator phases or between rotor phases do not vary; also the self-inductances are constant.

The resistance tensor is given by

$$R_{ij} = \begin{array}{c|ccccccc|c} & q^1 & q^2 & q^3 & q^4 & q^5 & q^6 & q^7 & \\ \hline & R_a & 0 & 0 & 0 & 0 & 0 & 0 & q^1 \\ & 0 & R_a & 0 & 0 & 0 & 0 & 0 & q^2 \\ & 0 & 0 & R_a & 0 & 0 & 0 & 0 & q^3 \\ i \downarrow & 0 & 0 & 0 & R_b & 0 & 0 & 0 & q^4 \\ & 0 & 0 & 0 & 0 & R_b & 0 & 0 & q^5 \\ & 0 & 0 & 0 & 0 & 0 & R_b & 0 & q^6 \\ & 0 & 0 & 0 & 0 & 0 & 0 & R_r & q^7 \\ \hline & & j \longrightarrow & & & & & & \end{array} \qquad (8.30)$$

where R_a is the resistance of a phase coil of the armature stator, R_b is the resistance of a rotor coil, and R_r is the resistance which represents the friction and windage loss.

The values of g_{ij} and R_{ij} as given by equations (8.29 and (8.30) could be substituted in equations (8.8) and the equations of performance obtained. The resulting equations are quite difficult to solve because some coefficients are trigonometric functions of the position of the rotor. If the two equations of constraint

$$\dot{q}^1 + \dot{q}^2 + \dot{q}^3 = 0 \qquad (8.31)$$

and

$$\dot{q}^4 + \dot{q}^5 + \dot{q}^6 = 0 \qquad (8.32)$$

are introduced, the equations can be simplified somewhat, but they are still quite involved. Levine (12) has given a treatment of this sort.

By introducing the linear transformation represented by equations (8.4) and (8.5), the voltages (including torque), resistances, inductances (including inertial coefficients), and currents can be referred to a set of orthogonal coordinate axes which are stationary with respect to the stator. The required transformation matrix is formed by applying equations (8.4) to both the stator and the rotor circuits. For the stator, the coefficients are constants; they are obtained by placing θ equal to zero degrees in equations. For the rotor, the coefficients are functions of θ; they are formed by changing the variables in equations (8.4) to the rotor variables of the induction motor. After the two non-singular conjugate transformation matrices have been obtained, the equation of constraint given by equation (8.32) is introduced to form the following two singular conjugate matrices:

$$\left[\frac{\partial q^j}{\partial s^i}\right] = \begin{array}{c} \begin{array}{cccccc} s^1 & s^2 & s^3 & s^4 & s^5 & \theta \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & +\frac{3}{2} & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{3}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & \cos(\theta+120) & \sin(\theta+120) & 0 \\ 0 & 0 & 0 & \cos(\theta-120) & \sin(\theta-120) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ q^1 \\ q^2 \\ q^3 \\ q^4 \\ q^5 \\ q^6 \\ \theta \end{array} \qquad (8.33)$$

$$\left[\frac{\partial s^i}{\partial q^j}\right] = \begin{array}{ccccccc|c} q^1 & q^2 & q^3 & q^4 & q^5 & q^6 & \theta & \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & s^1 \\ -\frac{1}{2} & \frac{3}{2} & 1 & 0 & 0 & 0 & 0 & s^2 \\ -\frac{1}{2} & -\frac{3}{2} & 1 & 0 & 0 & 0 & 0 & s^3 \\ 0 & 0 & 0 & \frac{2}{3}\cos\theta & \frac{2}{3}\cos(\theta+120) & \frac{2}{3}\sin(\theta-120) & 0 & s^4 \\ 0 & 0 & 0 & \frac{2}{3}\sin\theta & \frac{2}{3}\sin(\theta+120) & \frac{2}{3}\sin(\theta+120) & 0 & s^5 \\ 0 & 0 & 0 & & 0 & 0 & 1 & \theta \end{array} \qquad (8.34)$$

There are seven possible independent applied voltages (including torque) for a wound-rotor induction machine, but only four are customary. A transformation of the voltages is given by

$$e'_i = e_\alpha \frac{\partial q^\alpha}{\partial s^i}, \qquad (8.35)$$

where e_α represents one of the applied voltages, and e'_i, one of the transformed voltages. Using this formula, the following new voltages are obtained:

$$e'_1 = e_1 - \frac{1}{2} e_2 - \frac{1}{2} e_3 \; ; \qquad (8.36)$$

$$e'_2 = 0 + \frac{3}{2} e_2 - \frac{3}{2} e_3 \; ; \qquad (8.37)$$

$$e'_3 = e_1 + e_2 + e_3 \; ; \qquad (8.38)$$

$$e'_7 = e_7. \qquad (8.39)$$

The resistance tensor must also be transformed. Its new components are given by

$$R'_{ij} = R_{\alpha\beta} \frac{\partial q^\alpha}{\partial s^i} \frac{\partial q^\beta}{\partial s^j}. \qquad (8.40)$$

When the values of the resistances as given by equation (8.30) are substituted in equation (8.40), the following set of components is obtained:

$$R_{ij} = \begin{array}{c} \\ i \downarrow \end{array} \begin{array}{cccccc|c} s^1 & s^2 & s^3 & s^4 & s^5 & \theta & \\ \hline \frac{3}{2}R_a & 0 & 0 & 0 & 0 & 0 & s^1 \\ 0 & \frac{3}{2}R_a & 0 & 0 & 0 & 0 & s^2 \\ 0 & 0 & 3R_a & 0 & 0 & 0 & s^3 \\ 0 & 0 & 0 & \frac{3}{2}R_b & 0 & 0 & s^4 \\ 0 & 0 & 0 & 0 & \frac{3}{2}R_b & 0 & s^5 \\ 0 & 0 & 0 & 0 & 0 & R_r & \theta \\ \hline & & j \longrightarrow & & & & \end{array} \qquad (8.41)$$

The inductance tensor must be transformed in the same way as was the resistance tensor, but there are many more non-zero components; therefore an expanded form of the transformation formula is quite helpful. Since

$$g'_{ij} = g_{\alpha\beta} \frac{\partial q}{\partial s^i} \frac{\partial q}{\partial s^j}, \qquad (8.42)$$

then it follows that

$$g'_{ij} = L_a \left(\frac{\partial q^1}{\partial s^i} \frac{\partial q^1}{\partial s^j} + \frac{\partial q^2}{\partial s^i} \frac{\partial q^2}{\partial s^j} + \frac{\partial q^3}{\partial s^i} \frac{\partial q^3}{\partial s^j} \right)$$

$$- 2M_a \left(\frac{\partial q^1}{\partial s^i} \frac{\partial q^2}{\partial s^j} + \frac{\partial q^1}{\partial s^i} \frac{\partial q^3}{\partial s^j} + \frac{\partial q^2}{\partial s^i} \frac{\partial q^3}{\partial s^j} \right)$$

$$
\begin{aligned}
&+ L_b \left(\frac{\partial q^4}{\partial s^i} \frac{\partial q^4}{\partial s^j} + \frac{\partial q^5}{\partial s^i} \frac{\partial q^5}{\partial s^j} + \frac{\partial q^6}{\partial s^i} \frac{\partial q^6}{\partial s^j} \right) \\
&- 2M_b \left(\frac{\partial q^4}{\partial s^i} \frac{\partial q^5}{\partial s^j} + \frac{\partial q^4}{\partial s^i} \frac{\partial q^6}{\partial s^j} + \frac{\partial q^5}{\partial s^i} \frac{\partial q^6}{\partial s^j} \right) \\
&+ 2M \cos \theta \left(\frac{\partial q^1}{\partial s^i} \frac{\partial q^2}{\partial s^j} + \frac{\partial q^2}{\partial s^i} \frac{\partial q^6}{\partial s^j} + \frac{\partial q^3}{\partial s^i} \frac{\partial q^6}{\partial s^j} \right) \\
&+ 2M \cos(\theta+120) \left(\frac{\partial q^1}{\partial s^i} \frac{\partial q^2}{\partial s^j} + \frac{\partial q^2}{\partial s^i} \frac{\partial q^6}{\partial s^j} + \frac{\partial q^3}{\partial s^i} \frac{\partial q^4}{\partial s^j} \right) \\
&+ 2M \cos(\theta-120) \left(\frac{\partial q^1}{\partial s^i} \frac{\partial q^2}{\partial s^j} + \frac{\partial q^2}{\partial s^i} \frac{\partial q^4}{\partial s^j} + \frac{\partial q^3}{\partial s^i} \frac{\partial q^5}{\partial s^j} \right) \\
&+ J_r \frac{\partial \theta^7}{\partial s^i} \frac{\partial \theta^7}{\partial s^j}
\end{aligned}
\tag{8.43}
$$

When the coefficients from (8.33) are substituted in equation (8.43) for all the possible combinations of i and j, the following array of components is obtained:

$$
g_{ij} = \begin{array}{c|cccccc|c}
 & s^1 & s^2 & s^3 & s^4 & s^5 & \theta & \\
\hline
 & \frac{3}{2}(L_a+M_a) & -\frac{\sqrt{3}}{2} M_a & -6\, M_a & \frac{9}{2} M & 0 & 0 & s^1 \\
i \downarrow & -\frac{\sqrt{3}}{2} M_a & \frac{3}{2}(L_a+M_a) & -\sqrt{3}\, M_a & 0 & \frac{9}{2} M & 0 & s^2 \\
 & -6\, M_a & -\sqrt{3}\, M_a & 3(L_a+2M_a) & 0 & 0 & 0 & s^3 \\
 & \frac{9}{2} M & 0 & 0 & \frac{3}{2}(L_b+M_b) & -\frac{\sqrt{3}}{2} M_b & 0 & s^4 \\
 & 0 & \frac{9}{2} M & 0 & -\frac{\sqrt{3}}{2} M_b & \frac{3}{2}(L_b+M_b) & 0 & s^5 \\
 & 0 & 0 & 0 & 0 & 0 & J_r & \theta \\
\hline
 & & & j \longrightarrow & & & &
\end{array}
\tag{8.44}
$$

Now that the new sets of components of the resistance and the inductance tensors are available it is necessary in order to obtain the performance equations to substitute the voltages and the transformation coefficients into equation (8.9), and to perform the indicated operations. Since none of the new components of either the inductance or the resistance tensors are functions of the coordinates, the labor involved in solving the performance equations is considerably reduced.

For k = 1, the equation of performance, making use of equation (8.9), is

$$\frac{3}{2} R_a \dot{s}^1 + \frac{3}{2}(L_a + M_a) \ddot{s}^1 - \sqrt{\frac{3}{2}} M_a \ddot{s}^2 - 6 M_a \ddot{s}^3$$

$$+ \frac{9}{2} M \ddot{s}^4 = e_1 - \frac{1}{2}(e_2 + e_2), \qquad (8.45)$$

or, by replacing one of the derivatives with respect to time by p,

$$\frac{3}{2} \left[R_a + (L_a + M_a) p \right] \dot{s}^1 - \sqrt{\frac{3}{2}} M_a\, p\, \dot{s}^2 - 6\, M_a\, p\, \dot{s}^3 + \frac{9}{2} M\, p\, \dot{s}^4$$

$$= e_1 - \frac{1}{2}(e_2 + e_3). \qquad (8.46)$$

For k = 2, and k = 3, respectively, the following results obtain in a similar way:

$$- \frac{3}{2} M_a\, p\, \dot{s}^1 + \frac{3}{2} \left[R_a + (L_a + M_a) p \right] \dot{s}^2 - \sqrt{3}\, M_a\, p\, \dot{s}^3 + \frac{9}{2} M\, p\, \dot{s}^5$$

$$= \frac{3}{2} (e_2 - e_3) \; ; \qquad (8.47)$$

$$- 6\, M_a\, p\, \dot{s}^1 - \sqrt{3}\, M_a\, p\, \dot{s}^2 + 3 \left[R_a + (L_a - 2M_a) p \right] \dot{s}^3$$

$$= e_1 + e_2 + e_3. \qquad (8.48)$$

It should be noted that for $k = 1, 2,$ and 3 that no terms in θ are involved. From all appearances, equations (8.46), (8.47), and (8.48) might be representing a stationary network.

There are three remaining equations to determine. Two of the three equations are electric circuit equations involving motional impedances, and the third is a mechanical circuit equation which represents the rotor mechanical effects.

For $k = 4$ and $g'_{\alpha\beta} = g'_{55}$ the torsion term

$$g'_{\alpha\beta}\left[\frac{\partial}{\partial q^m}\frac{\partial s}{\partial q^i} - \frac{\partial}{\partial q^i}\frac{\partial s}{\partial q^m}\right]\frac{\partial q^i}{\partial s^k}\frac{\partial q^m}{\partial s}\dot{s}^\alpha \quad \dot{s}^\gamma$$

yields

$$g'_{55}\left[\frac{\partial}{\partial\theta}\left(\frac{2}{3}\sin\theta\right)\cos\theta + \frac{\partial}{\partial\theta}\left(\frac{2}{3}\sin(\theta+120)\right)\cos(\theta+120)\right.$$
$$\left. + \frac{\partial}{\partial\theta}\left(\frac{2}{3}\sin(\theta-120)\right)\cos(\theta-120)\right]\dot{\theta}\,\dot{s}^5 =$$
$$\frac{3}{2}(L_b + M_b)\,\dot{\theta}\,\dot{s}^5 \tag{8.49}$$

Similarly, for $k = 4$ and $g'_{\alpha\beta} = g'_{25}$, the torsion term yields

$$g'_{25}\left[\frac{\partial}{\partial\theta}\left(\frac{2}{3}\sin\theta\right)\cos\theta + \frac{\partial}{\partial\theta}\frac{2}{3}\sin(\theta+120)\cos(\theta+120)\right.$$
$$\left. + \frac{\partial}{\partial\theta}\left(\frac{2}{3}\sin(\theta-120)\right)\cos(\theta-120)\right]\dot{\theta}\,\dot{s}^2 =$$
$$\frac{9}{2}M\,\dot{\theta}\,\dot{s}^2. \tag{8.50}$$

For all other values of $g'_{\alpha\beta}$ and $k = 4$, the torsion terms vanish; therefore the equation of performance for $k = 4$ can be written in the form

$$\frac{9}{2} M\ p\ \dot{s}^1 + \frac{3}{2}\Big[R_b + (L_b + M_b)p\Big]\dot{s}^4 + \sqrt{\frac{3}{2}}\, M_b p\ \dot{s}^5$$

$$+ \frac{3}{2}(L_b + M_b)\ \dot{\theta}\ \dot{s}^5 + \frac{9}{2} M\ \dot{\theta}\ \dot{s}^2 + 0 \qquad (8.51)$$

Similarly for $k = 5$, the following equation is obtained:

$$\frac{9}{2} M\ p\ \dot{s}^2 - \sqrt{\frac{3}{2}}\, M_b\ p\ \dot{s}^4 + \frac{3}{2}\Big[R_b + (L_b + M_b)p\Big]\dot{s}^5$$

$$- \frac{3}{2}(L_b + M_b)\ \dot{\theta}\ \dot{s}^4 - \frac{9}{2} M\ \dot{\theta}\ \dot{s}^1 = 0. \qquad (8.52)$$

It now is necessary to calculate the terms of the torque equation; that is to say, the equation of performance for $k = 7$. This particular equation has the following simple form:

$$(\ R_r + J_r\ p)\ \dot{\theta} = e_7 \qquad (8.53)$$

The quantity e_7, it should be remembered, is the applied torque on the rotor.

The complete set of six differential equations are collected below:

$$\frac{3}{2}\Big[R_a + (L_a + M_a)p\Big]\dot{s}^1 - \sqrt{\frac{3}{2}}\, M_a\ p\ \dot{s}^2 - 6\ M_a\ p\ \dot{s}^3 + \frac{9}{2} M\ p\ \dot{s}^4 =$$

$$e_1 - \frac{1}{2}(e_2 + e_3) \qquad (8.46)$$

$$- \sqrt{\frac{3}{2}}\, M_a\ p\ \dot{s}^1 + \frac{3}{2}\Big[R_a + (L_a + M_a)p\Big]\dot{s}^2 - \sqrt{3}\, M_a\ p\ \dot{s}^3 + \frac{9}{2} M\ p\ \dot{s}^5 =$$

$$\frac{3}{2}(e_2 - e_3) \qquad (8.47)$$

$$- 6\ M_a\ p\ \dot{s}^1 - \sqrt{3}\, M_a\ p\ \dot{s}^2 + 3\Big[R_a + (L_a - 2M_a)p\Big]\dot{s}^3 =$$

$$e_1 + e_2 + e_3. \qquad (8.48)$$

$$\frac{9}{2} M\, p\, \dot{s}^1 + \frac{3}{2}\left[R_b + (L_b + M_b)p\right]\dot{s}^4 + \sqrt{\frac{3}{2}} M\, p\, \dot{s}^5$$

$$+ \frac{3}{2}(L_b + M_b)\, \dot{\theta}\, \dot{s}^5 + \frac{9}{2} M\, \dot{\theta}\, \dot{s}^2 = 0 \qquad (8.51)$$

$$\frac{9}{2} M\, p\, \dot{s}^2 - \sqrt{\frac{3}{2}} M_b\, p\, \dot{s}^4 + \frac{3}{2}\left[R_b + (L_b + M_b)p\right]\dot{s}^5$$

$$- \frac{3}{2}(L_b + M_b)\, \dot{\theta}\, \dot{s}^4 - \frac{9}{2} M\, \dot{\theta}\, \dot{s}^1 = 0 \qquad (8.52)$$

$$(R_r + J_r\, p)\, \dot{\theta} = e_7 \qquad (8.53)$$

These equations represent the complete performance, both transient and steady-state, of the induction machine. It is necessary, however, in order to obtain a complete solution of these equations to specify either applied voltages, including the rotor torque, and suitable initial conditions, or applied currents, including the rotor velocity, and suitable initial conditions.

The equations might be given a sort of physical interpretation. The stator currents have been projected on orthogonal axes, two of which are along and perpendicular to, respectively, the magnetic axis of phase 1, and the third of which might be considered perpendicular to the plane of the other two. A separate and independent coil with various mutual-inductive couplings might be associated with each of these axes. To each of these coils a different voltage would be applied. Rotating inside of this set of three coils, another set of three coils could be pictured. These rotating coils should have mutual-inductive couplings between themselves and between the stationary coils; also they should be pictured as connected to commutator bars with two external brush-connections, arranged stationary in space, along and perpendicular to, respectively, the magnetic axis of the stator phase 1. This picture in some ways resembles that of a d-c machine.

Usually the three voltages applied to an induction machine are balanced, that is, their instantaneous sum is equal to zero. Also, usually only three wires lead to the machine; therefore some simplifications in the equations can be made. The sum of the applied voltages being zero makes it possible to eliminate from the equations one of the three applied voltages; only three wires existing makes it necessary that the sum of the three line currents be zero. If the sum of the line currents is equal to zero, another equation is eliminated; namely, equation (8.48). There remain only five simultaneous equations to solve. These remaining five equations are obtained by placing $\dot{s}^3$ equal to zero in equations (8.46) to (8.53) and by dropping equation (8.48). These statements are based upon the mathematics involved.

One group of transient conditions which might be treated by this method consists of that for which the rotor speed is constant. In this case Θ can be replaced by a constant; then the set of equations can be solved by Heaviside or Laplacian-transform methods, just as any stationary network.

The same method which has been used here may also be applied to two-phase and single-phase induction motors. Of course the number of simultaneous equations involved will be less than those required for the three-phase case.

The results obtained by the method given are similar in many respects to those given by Stanley (21). Stanley used a non-invariant type of transformation; that is, Stanley transformed all quantities by means of the same set of transformation coefficients without using the conjugate set, so that in his analysis the expression for power did not maintain the invariant form

$$e_i \, i^i = e'_i \, i^i.$$

His methods were not those of tensor analysis but of an even less restricted class from matrix theory. As a result his performance equations were somewhat simpler in form than those given here.

It is interesting to note that the performance equations given might be used with constant rotor speed and also with three-phase sinusoidal voltages in order to obtain the steady-state performance. In this case, various reactances could be defined and an equivalent circuit could be devised.

Special consideration might be given to the case of pulsating voltage and torque variations to obtain some very interesting equations, but this special consideration will not be taken up here.

The method given here is not limited to the determination of the steady-state performance of machines. The usual methods of analyzing machines are based on some sort of equivalent electrical circuit which happens to represent the machine for some specific steady-state conditions, but they are not suited for the determination of transient conditions. Methods for determining transient performance of machines seem to be of increasing interest to the engineering profession as design limitations are often determined by transient conditions. It is desirable therefore that an engineer have among his methods of attack upon engineering problems some method for the determination of transient conditions of electrical machinery. The method given here has an additional advantage in that it is based on a unification of electrical and mechanical systems. Besides, the method can be applied with little change to other problems such as electro-acoustical problems.

IX. SUMMARY AND CONCLUSIONS

The first five parts of this paper are devoted to the fundamentals of tensor analysis, both the usual and the intrinsic, and the application of these fundamentals to differential geometry. Many of the concepts of tensor theory are introduced in the treatment of differential geometry. This procedure was followed in order to take advantage of the geometrical method of reasoning. After a brief introduction, the fundamental definitions and algebraic manipulations of tensors in themselves without reference to any geometrical space are considered. Also pseudo-tensors are introduced and related to true tensors. Then, under the heading of differential geometry, a short treatment of non-metrical geometry is included. The true change in a tensor component due to a small change in coordinates or intrinsic variables is defined. This definition leads to the definition of the covariant and the intrinsic derivatives with respect to a symmetrical affine connection with an undefined metric, the choice of the metric being left open. Later, a metric is introduced by means of the fundamental quadratic form $ds^2 = g_{ij}\, dx^i\, dx^j$ which converts the hitherto amorphous space into a metrical space. After the introduction of the metric, many of the previously introduced concepts assume special forms; for example, the symmetric affine connection becomes a three-index, Christoffel symbol. More consideration is given to the special Riemannian metric geometry, that is, the metric geometry with gauge invariance and a symmetric connection, than to the non-Riemannian metric geometry, that is, the metric geometry with an untransportable metric, except in an infinitesimal region, and a general connection. The invariant methods of the intrinsic tensor analysis in general require a general connection; therefore a non-Riemannian metric geometry may be defined by the concepts of the intrinsic tensor analysis, in connection with the fundamental quadratic form.

In part VI of this paper, some of the concepts of the usual vector analysis are related to tensor analysis by making use of the pseudo-vector representation of second order anti-symmetric tensors. Such concepts as the gradient, the divergence, the curl, the Laplacian, and Stoke's theorem were given consideration.

In part VII, some of the important analytical concepts of tensor dynamics are given. The method of procedure is based on the initial postulate that the absolute or intrinsic derivative of the generalized momentum is equal to the external forces. By expanding the intrinsic derivative and by introducing the concept of kinetic energy, Lagrange's equations are transformed from one coordinate system to another, and from a coordinate system to an ennuple of congruences. Both holonomic and non-holonomic constrained dynamical systems are discussed.

Part VIII of this paper is concerned with electro-mechanical systems, chiefly the rotating electrical machine. The more important analogous concepts connected with electrical and mechanical systems are tabulated. The dynamical equations of motion expressed in terms of coordinate variables and intrinsic variables are assumed to represent the performance of electro-mechanical systems. The method of analysis requires the knowledge of certain electrical and mechanical design constants (inductances, inertial coefficients, resistances, etc.) of the system under consideration. A general method of analyzing rotating electrical machines, including the type of variation of the electrical design constants, is given. Then the general method is applied to the induction machine. As the performance equations in terms of Riemannian coordinates contain functional coefficients, a previously introduced type of transformation which refers the system to a special set of non-Riemannian coordinates is made. This transformation results in an equivalent set of performance equations which contain only cyclic variables. These equivalent performance equations can be solved quite easily for most cases. In the case of constant rotor speed the equations assume an especially simple form, involving constant coefficients, in terms of the electric currents and their rates of change; therefore either the Heaviside operational methods or the Laplacian-transform methods of solving linear differential equations with constant coefficients can be applied.

Experience has demonstrated that in practical applications the non-invariant type of transformations is more flexible and less restricted than are the invariant transformations in which certain linear or multilinear algebraic forms are maintained invariant as is the case of tensor analysis. In the application of tensor methods to the induction motor the fundamental quadratic form $ds^2 = g_{ij}\, dx^i\, dx^j$ and the linear form $e_i\, i^i$ are two examples of invariant forms. By comparing the results of the author's invariant transformation method of solving the induction motor with the results of Stanley's (21) non-invariant method, it is clear that Stanley's non-invarient transformations yielded a more simple group of performance equations.

Future work in this field might include numerous applications of the methods given here or an expansion of these methods to include the theory of groups, topology advanced matrix and determinant theory, and further generalized geometries. As revolving waves are involved in the physical pictures of the mathematics of rotating electrical machines, and as Schrodinger's wave mechanics follows from advanced dynamical methods, it seems that perhaps the methods of Schrodinger's wave mechanics might be applied to rotating electrical machinery.

X. LITERATURE CITED

1. Blondel, André. Methods of calculation of the armature reaction of alternators. Papers of the International Electrical Congress. St. Louis. 1:635. 1904.

2. Brillouin, Leon. Les tenseurs en mécanique et en elasticite. Masson et Cie. Paris. 1938.

3. Cartan, E. Sur les variétiés à connexion affine et la théorie de la relativité généralisée, Ann., École Norm. 3e serie. 40:325-412, 1923; 41:1-25, 1924; 42:17-88, 1925.

4. Christoffel, E. B. Über die Transformation der homogenen Differential - ausdrucke sweiten Grades. Jour. fur die reine und angew. Math, (Crelle) 70:46-70. 1876.

5. Eddington, A. S. The mathematical theory of relativity. Cambridge University Press. Cambridge. 1923.

6. Einstein, A. The foundation of the general theory of relativity. In Lorentz, H. A. The principle of relativity. pp. 109-164. Methuen, London. 1923.

7. Einstein, A. The meaning of relativity. Princeton University Press. Princeton. 1923.

8. Graustein, W. C. The geometry of Riemannian spaces. Trans. American Math. Soc. 36:542-585. 1934.

9. Kron, Gabriel. Non-Riemannian dynamics of rotating electrical machinery. Jour. of Math. and Physics. 13:103-194. 1934

10. Kron, Gabriel. The application of tensors to the analysis of rotating machinery. General Electric Review. Schenectady. 1938.

11. Lagrange, J. L. Mécanique analytique. In his Oeuvres. Vol. 11-12 Gauthier-Villars, Paris. 1888.

12. Levine, S. J. An analysis of the induction motor. Trans. A.I.E.E. 54:526-529. 1935.

13. Lindsay, R. B., Margenau, H. Foundations of physics, pp. 59-72. John Wiley, New York. 1936.

14. Lipshitz, R. Ausdehnung der Theorie der Minimalflachen. Jour. fur die reine und angew. Math. (Crelle). 78:1-45. 1874.

15. Murnaghan, F. D. Vector analysis and the theory of relativity. Johns Hopkins Press, Baltimore. 1922.

16. Park, R. H. Two-reaction theory of synchronous machines. Trans. A.I.E.E. 48:716-730. 1929.

17. Prentice, B. R. Fundamental concepts of synchronous machine reactances. Trans. A.I.E.E. Vol. 57, supplement. 1937.

18. Ricci, G., and Levi-Civita, T. Méthods de calcul différential absolu et leurs applications. Mathematische Annalen. 54:125-201, 608. 1901.

19. Riemann, B. Über die Hypothesen, welche der Geometrie zu Grunde liegen. Gesammelte Math. Werke. 2nd edition, pp. 272-288. Teubner, Leipzig. 1892.

20. Schouten, J. A. Die direkte Analysis zur neuren Relativitätstheorie. Verhandelingen Kon. Akad. Amsterdam. Vol. 12, No. 6. 1918.

21. Stanley, H. C. An analysis of the induction motor. Trans. A.I.E.E. Vol. 57, supplement. 1938.

22. Veblen, Oswald. Invariants of quadratic differential forms. Cambridge Tract in Math. and Math. Physics. No. 24. Cambridge University Press, Cambridge. 1927.

23. Vranceanu, M. G. Les espaces non holonomes et leur applications mécaniques. Mémorial des Sciences Mathématiques. No. 76. Gauthier-Villars, Paris. 1936.

24. Weyl, H. Reine Infinitesimalgeometrie. Math. Zeits. 2:384-411. 1918.

25. Weyl, H. Gravitation and electricity. In Lorentz, H. A. The principle of relativity. pp. 199-216. Methuen, London. 1923.

26. Whittaker, E. T. A treatise on the analytical dynamics of particles and rigid bodies. p. 43. Cambridge University Press, Cambridge. 1927.

XI. ACKNOWLEDGMENTS

The author hereby acknowledges his indebtedness and expresses his sincere appreciation to Dr. H. W. Anderson under whose direction this work was accomplished; to Professor M. S. Coover for his helpful counsel; to Dr. W. A. Thomas for his frank criticism; to those staff members of the electrical engineering, physics, and mathematics departments who have given aid to the author; and to those men, living and dead, who have developed tensor analysis and related subjects and upon whose works the author has so freely drawn.